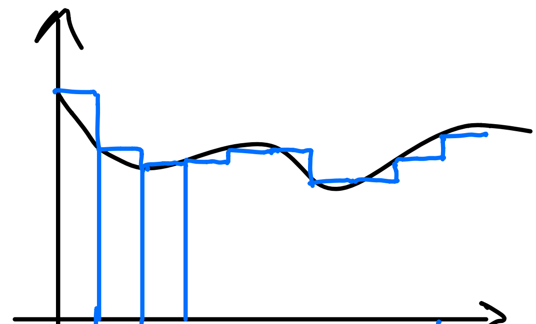
Session 16
Oct. 26, 2018

3.2 Stochastic Integrals

Recall Riemann sum for Riemann integral:

$$\int_0^T f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i$$



$$\Delta t_i = t_{i+1} - t_i = \Delta t = \frac{T}{n}$$

later: want stochastic $\underbrace{\text{PDEs}}_{\text{partial differential equation}}$ with noise: $dX = f\,dt + g\,dW$

there are different kinds of stochastic integrals

Itô-integral:

def. analogously to Riemann sum (W = Brownian motion)

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \quad \text{with} \quad \Delta W_i = W(t_{i+1}) - W(t_i) \sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$$

Ex.: integrate Brownian motion against itself: $\int_0^T W(t)\,dW(t) = \int_0^T W\,dW$

• W is not differentiable: $\frac{d}{dt} f(g(t)) = f' \cdot \frac{dg}{dt}$

Cannot use $dW = \underbrace{\frac{dW}{dt}}_{\text{doesn't exist (BM not differentiable)}} dt$

cannot write $\int_0^T W(t)\,dW(t) = \int_0^T W(t) \frac{dW(t)}{dt} dt = \frac{1}{2} \int_0^T \frac{d}{dt}(W(t)^2)\,dt$

$$= \frac{1}{2} W(T)^2 - \frac{1}{2} \underbrace{W(0)^2}_{=0}$$

value of the integral is actually different

• $\int_0^T W(t)\,dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)\Delta W_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i)\left(W(t_{i+1}) - W(t_i)\right)}$

$$= W(t_i)W(t_{i+1}) - W(t_i)^2$$

$$= \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 - \left(W(t_{i+1}) - W(t_i)\right)^2\right]$$

$$\Rightarrow \int_0^T W(t)\,dW(t) = \underbrace{\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2\right]}_{= \frac{1}{2} W(T)^2 - \frac{1}{2} \underbrace{W(0)^2}_{=0}} - \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(W(t_{i+1}) - W(t_i)\right)^2$$

How is $\left[W(t_{i+1}) - W(t_i)\right]^2 = \Delta W_i^2$ distributed

It turns out that • $\mathbb{E}(\Delta W_i^2) = \Delta t = \frac{T}{n}$

$$\bullet\ \mathbb{E}(\Delta W_i^4) = \Delta t^2 = \frac{T^2}{n^2}$$

so $\sum_{i=0}^{n-1} \Delta W_i^2 = \sum_{i=0}^{n-1} \left(\frac{T}{n} + O\left(\frac{1}{n^2}\right)\right) = T + O\left(\frac{1}{n}\right) \xrightarrow{n\to\infty} T$

$\Rightarrow$ in the limit $n\to\infty$, $\sum_{i=0}^{n-1} (\Delta W_i)^2$ is const (deterministic process)

$$\Rightarrow \int_0^T W(t)\,dW(t) = \frac{1}{2} W(T)^2 - \frac{1}{2} T$$

(different from usual integral because $\Delta W \sim \sqrt{\Delta t}$ and not like Δt)

<u>Stratonovich integral:</u>

$$\int_0^T f(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i^*)\,\Delta W_i \quad \text{with } t_i^* = \frac{t_{i+1}+t_i}{2}$$

<u>Ex.:</u> $\int_0^T W(t)\circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i^*)\left(W(t_{i+1}) - W(t_i)\right)}$

$$\frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 + \left(W(t_i^*) - W(t_i)\right)^2 - \left(W(t_{i+1}) - W(t_i^*)\right)^2 \right]$$

$$\Rightarrow \int_0^T W(t)\circ dW(t) = \frac{1}{2} W(T)^2 + \lim_{n\to\infty}\left[\sum_{i=0}^{n-1} \left(W(t_i^*) - W(t_i)\right)^2 - \sum_{i=0}^{n-1}\left(W(t_{i+1}) - W(t_i^*)\right)^2\right]$$

similar to before: $\mathbb{E}\left((W(t_i^*)^2 - W(t_i))^2\right) \sim t_i^* - t_i = \frac{t_{i+1}+t_i}{2} - t_i$

$$= \frac{t_{i+1}-t_i}{2} = \frac{\Delta t}{2}$$

and higher moments vanish if summed over similar to above

$$\Rightarrow \int_0^T W(t) \circ dW(t) = \frac{1}{2} W(T)^2 + \frac{T}{2} - \frac{T}{2} = \frac{1}{2} W(T)^2$$

In comparison:

- Stratonovich:
  - some "nicer" properties and better analogy to usual integral
  - but in each step W is evaluated in between t_i and t_{i+1}
- Itô:
  - technically a bit "harder" to handle
  - but at t_i, the increments ΔW_i are added, as we want for stock price development

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Summary of the course so far:

Options: calls, put; American, European } vanilla options

(note: many other types of options: Asian, Bermudan, Exotic, ...)

here: options on Stocks (also: options on ressources, currencies, interest rates, ...)

need two things: (1) model for stock prices

(2) price options fairly (no arbitrage) using such models by considering replicating portfolios

- discrete time models:

  binomial tree stock price model
$$\begin{tikzcd} & Su \\ S \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & Sd \end{tikzcd}$$

  ↳ pricing by backward induction

  ↳ advantage: very versatile (dividends, discrete interest compounding etc. can easily be implemented)

  ↳ special case of European calls:

  ⇒ closed formula: $C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, Su^j d^{n-j} - K)$

  $= e^{-rT}\, \mathbb{E}(\text{payoff})$ w.r.t. binomial dist.

  ⇒ in the limit $n \to \infty$ this becomes Black-Scholes formula:

$$C = S\Phi(x) - Ke^{-rT}\Phi(x - \sigma\sqrt{T}), \quad x = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

  Φ = cumulative normal distribution fct.

- continuous time models:

  geometric Brownian Motion stock price model:

  ↳ next: pricing using GBM

- any model:

this is actually a deep and general result based on replicating portfolio

If price $C = e^{-rT}\,\mathbb{E}(\ldots)$, then Monte-Carlo method gives us a good and fast approximation.