# 3.3 Stochastic Differential Equations

Session 17
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usual first order ordinary differential equation (ODE):

$$\frac{dx(t)}{dt} = f(x(t), t)$$

integral form: $x(t) = x(0) + \int_0^t f(x(s), s)\, ds$

stochastic version (SDE):

$$X(t) = X(0) + \int_0^t f(X(s), s)\, ds + \underbrace{\int_0^t g(X(s), s)\, dW(s)}_{\text{stochastic integral (always Itô from now on)}}$$

→ Brownian motion increments

short-hand notation: $dX(t) = f(X(t), t)\, dt + g(X(t), t)\, dW(t)$

Ex.: $dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t)$ , $S(0) = S_0$

next time: this is solved by geom. Brownian motion

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

integral form: $S(t) - S_0 = \mu \int_0^t S(u)\, du + \sigma \int_0^t S(u)\, dW(u)$

from this we can compute $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u))\,du + \sigma \int_0^t \underbrace{\mathbb{E}(S(u)\,dW(u))}_{=\mathbb{E}(S(u))\underbrace{\mathbb{E}(dW(u))}_{=0}}$$

so

$$\mathbb{E}(S(t)) = S_0 + \mu \int_0^t \mathbb{E}(S(u))\,du$$

$$\left( \frac{d\,\mathbb{E}(S(t))}{dt} = \mu\, \mathbb{E}(S(t)) \right)$$

$$\Rightarrow \mathbb{E}(S(t)) = e^{\mu t}$$

Usual ODE can be solved numerically with Euler's method:

discretize ODE: $\frac{x_{n+1} - x_n}{\Delta t} = f(x_n, t_n)$ , $\Delta t = \frac{t}{n}$

or $x_{n+1} = x_n + f(x_n, t)\Delta t$

error in one step:

- Euler: $x_1 = x_0 + f(x_0, t)\Delta t$
- exact solution (with Taylor): $x(\Delta t) = x(0) + \Delta t \overbrace{x'(0)}^{=f(x_0,0)} + \frac{(\Delta t)^2}{2} x''(0) + \mathcal{O}((\Delta t)^3)$

$$\Rightarrow x(\Delta t) - x_1 = \frac{(\Delta t)^2}{2} x''(0) + \mathcal{O}((\Delta t)^3) \approx c\,(\Delta t)^2$$

total error: $|x(t) - x_n| \sim c(t)\,\Delta t \qquad , \Delta t = \frac{t}{n}$

the generalization to SDEs is called Euler-Maruyama method:

$$X_{n+1} = X_n + f(X_n, t_n)\,\Delta t + g(X_n, t_n)\,\Delta W_n$$

for error we want to compare $X_n$ to exact sol. $X(t)$.

one distinguishes two types of errors:

- strong error: $\mathbb{E}(|X_n - X(t)|) \sim c_s (\Delta t)^\alpha$

  $\alpha$ = strong order of convergence

note: relevance for individual paths via Markov's inequality

$$\mathbb{P}(|X| > a) \leq \frac{\mathbb{E}(|X|)}{a}$$

Proof:

$$\mathbb{E}(|X|) = \int_{-\infty}^{\infty} |x| \underbrace{\rho(x)}_{\text{probability density}} dx = \int_{-a}^{a} |x|\rho(x)dx + \int_{-\infty}^{-a} |x|\rho(x)dx + \int_{a}^{\infty} |x|\rho(x)dx$$

$$\geq \int_{-\infty}^{-a} |x|\rho(x)\,dx + \int_{a}^{\infty} \underbrace{|x|}_{>a}\rho(x)\,dx$$

use $|x| > a$ integral

$$\geq a \underbrace{\left( \int_{-\infty}^{-a} \rho(x)\,dx + \int_{a}^{\infty} \rho(x)\,dx \right)}_{\mathbb{P}(|X| > a)}$$

Then in our case:

$$\Rightarrow \mathbb{P}\left(|X_n - X(t)| > (\Delta t)^{\frac{\alpha}{2}}\right) \le \frac{c_s (\Delta t)^{\alpha}}{(\Delta t)^{\frac{\alpha}{2}}} = c_s (\Delta t)^{\frac{\alpha}{2}}$$

probability of a large error small for individual paths

- weak error: $|\mathbb{E}(X_n) - \mathbb{E}(X(t))| \sim c_w (\Delta t)^{\beta}$, $\beta$ = weak order of convergence

note: $|\mathbb{E}(X_n) - \mathbb{E}(X(t))| = |\mathbb{E}(X_n - X(t))|$

$$\le |\mathbb{E}(|X_n - X(t)|)|$$

so weak error $\le$ strong error

Ex.: compare $X(t) = 0$ to $X_n = \begin{cases} +1 & \text{with prob. } \frac{1}{2} \\ -1 & \text{with prob. } \frac{1}{2} \end{cases}$

weak error: $|0 - 0| = 0$

strong error: $\frac{1}{2} + \frac{1}{2} = 1$