3.4 Itô-lemma

Nov. 9, 2018

first version: consider (nice) fct. $h(W(t), t)$.

Goal: find a stochastic version of the chain rule

first, look at $h = h(W(t))$ (meaning $\frac{\partial h}{\partial t} = 0$)

write $h(W(t)) - h(W(0)) = \sum_{j=0}^{n-1} \Big(h(W(t_{j+1})) - h(W(t_j)) \Big)$

Taylor expansion:

$$h(W(t)) - h(W(0)) = \sum_{j=0}^{n-1} h'(W(t_j)) \Big(W(t_{j+1}) - W(t_j) \Big)$$
$$+ \sum_{j=0}^{n-1} \frac{1}{2} h''(m_j) \Big(W(t_{j+1}) - W(t_j) \Big)^2$$

for some $m_j = W_{s_j}$ with $s_j \in [t_j, t_{j+1}]$

now: recall $W(t_{j+1}) - W(t_j) \sim \sqrt{\Delta t}\, \mathcal{N}(0,1)$

and as before $\Big(W(t_{j+1}) - W(t_j)\Big)^2 \xrightarrow{n \to \infty} dt$

take $\lim_{n\to\infty}$: $h(W(t)) - h(W(0)) = \int_0^t \left(\frac{\partial h}{\partial x}\right)(W(s))\, dW(s)$

$$+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 h}{\partial x^2}\right)(W(s))\, ds$$

$\Rightarrow$ in general case where $h(W(t), t)$ we have the Itô formula:

$$h(W(t),t) - h(W(0),0) = \int_0^t \left(\frac{\partial h}{\partial x}\right)(W(s),s)\, dW(s)$$
$$+ \int_0^t \left[\left(\frac{\partial h}{\partial s}\right)(W(s),s) + \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)(W(s),s)\right] ds$$

short-hand notation: $dh = h' dW + \dot{h} dt + \frac{1}{2} h'' dt$

$\left(\text{here: } h' = \frac{\partial h}{\partial x},\ \dot{h} = \frac{\partial h}{\partial t}\right)$

Ex.:

- $h(W(t),t) = W(t)^2$

  Itô: $dh = 2W dW + 0 + \frac{1}{2} 2\, dt = 2W dW + dt$

  is the SDE with solution $h = W^2$

  $$\Rightarrow h(W(t)) - \underbrace{h(W(0))}_{=0} = \int_0^t 2W(s)\, dW(s) + \int_0^t ds$$

  e.g., $\mathbb{E}\big(h(W(t))\big) = \mathbb{E}(W(t)^2) = \int_0^t 2\, \mathbb{E}(W(s))\, \underbrace{\mathbb{E}(dW(s))}_{=0} + t$

- $h(W(t),t) = W(t)^4$

  $\Rightarrow dh = 4W^3 dW + 6W^2 dt$

  e.g., $\mathbb{E}(W(t)^4) = 4\int_0^t \mathbb{E}(W(s)^3)\, \mathbb{E}(dW(s)) + 6\int_0^t \mathbb{E}(W(s)^2)\, ds$

$$= \quad 0 \qquad + 6\int_0^t s\,ds$$

$$= 3t^2$$

Ex.: solve $dX = X^3 dt - X^2 dW$, $X(0)=1$

write $X = h(W(t),t)$ and compare $dX = dh$

Itô: $dh = h' dW + \dot{h} dt + \frac{1}{2} h'' dt$

$\Rightarrow$ need to solve $\dot{h} + \frac{1}{2}h'' = h^3$ and $h' = -h^2$

$$\frac{dh}{dx} = -h^2 \overset{\substack{\text{separation} \\ \text{of variables}}}{\Longrightarrow} \frac{dh}{-h^2} = dx \Rightarrow \int \frac{dh}{-h^2} = \int dx$$

$$\Rightarrow \frac{1}{h} = x + C$$

$$\Rightarrow h(x,t) = \frac{1}{x+C}$$ with $h(W(0),0) = h(0,0) = 1$ (initial condition)

$$\Rightarrow h(x,t) = \frac{1}{x+1}$$ (independent of t)

$\left(\text{check: } \dot{h} + \frac{1}{2}h'' = 0 + \frac{1}{2}\frac{2}{(x+1)^3} = \frac{1}{(x+1)^3} = h^3 \checkmark \right)$

$$\Rightarrow \text{solution } X(t) = \frac{1}{W(t)+1}$$ (note: actually blows up in finite time)

Second version: consider $dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dW(t)$

this is called an Itô process

now consider (nice) fct. $F(X(t), t)$

informally: Taylor expansion:

$$\Delta F(x,t) = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial x}\underbrace{\Delta x}_{= f\Delta t + g\Delta w} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Delta x^2 + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 F}{\partial x \partial t}\Delta x \Delta t}_{\text{neglect, lower order}}$$

$$(\Delta x)^2 = (f\Delta t + g\Delta w)^2 = \underbrace{f^2\Delta t^2 + fg\,\Delta t\Delta w}_{\text{neglect, lower order}} + g^2 \underbrace{\Delta w^2}_{\sim \Delta t}$$

$$\Rightarrow \Delta F = \frac{\partial F}{\partial t}\Delta t + f\frac{\partial F}{\partial x}\Delta t + g\frac{\partial F}{\partial x}\Delta W + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}g^2\Delta t$$

<u>Itô-lemma:</u>

$$dF = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}g^2\right]dt + g\frac{\partial F}{\partial x}dW$$

note: for $X(t) = W(t)$, i.e., $f=0$, $g=1$, we get

$$dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right]dt + \frac{\partial F}{\partial x}dW$$

(i.e., reduces to Itô formula from above in this special case)

<u>Ex.:</u> geometric Brownian motion $S(W(t),t) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$

$\Rightarrow$ corresponding SDE is $dS = \left[\left(\mu - \frac{\sigma^2}{2}\right)S + \frac{1}{2}\sigma^2 S\right]dt + \sigma S dW$

$$\Rightarrow dS = \mu S dt + \sigma S dW$$

What is $\mathbb{E}(S(t)^n)$?

Write $F(S(t), t) = S(t)^n$

$$dS^n = \left[\mu S n S^{n-1} + \frac{1}{2}\sigma^2 S^2 n(n-1) S^{n-2}\right]dt + \sigma S n S^{n-1} dW$$

$$= S^n\left(\mu n + \frac{1}{2}\sigma^2 n(n-1)\right)dt + n\sigma S^n dW$$

$$\Rightarrow \mathbb{E}(S(t)^n) - \mathbb{E}(S_0^n)$$

$$= \left(n\mu + \frac{1}{2}\sigma^2 n(n-1)\right)\int_0^t \mathbb{E}(S(t)^n)dt + n\sigma \int_0^t \mathbb{E}(S(t)^n)\underbrace{\mathbb{E}(dW(s))}_{=0}$$

$$\Rightarrow \mathbb{E}(S(t)^n) = S_0^n e^{(n\mu + \frac{1}{2}n(n-1)\sigma^2)t}$$

in particular: $\bullet\ \mathbb{E}(S(t)) = S_0 e^{\mu t}$

$$\bullet\ Var(S(t)) = \mathbb{E}(S(t)^2) - \mathbb{E}(S(t))^2$$

$$= S_0^2 e^{(2\mu + \sigma^2)t} - S_0^2 e^{2\mu t}$$

$$= S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

HW: $X =$ geom. BM , $F(X,t) = (1+t)\sqrt{X}$

Verify Itô's lemma numerically

- find SDE for F

- solve it numerically with Euler-Maruyama

- compare with given exact solution