# 4. Black-Scholes Equation and Finite Difference Schemes

Session 22
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## 4.1 Derivation of Black-Scholes Equation

Let the stock price process be geom. BM: $dS = \mu S\,dt + \sigma S\,dW$

solution: $S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$

Let $C(S,t)$ be the price of an option

Itô's Lemma gives $dC = \left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S}\sigma S\,dW$

(recall Itô: $dX = f\,dt + g\,dW$ then $dF(X,t) = \left( \frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2} \right) dt + \frac{\partial F}{\partial x} g\,dW$ )

Merton's trick: consider the right portfolio to eliminate risk

value of portfolio $\Pi = -C + \frac{\partial C}{\partial S} S$

$$d\Pi = -dC + \frac{\partial C}{\partial S}dS = r\Pi\,dt \quad // \quad \boxed{r\left(-C + \frac{\partial C}{\partial S}S\right)dt}$$

(such that $\Pi = \Pi_0 e^{rt}$)

such a portfolio grows with riskless rate $r$ (deterministically)

$$= -\left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt - \frac{\partial C}{\partial S}\sigma S\,dW$$

$$+ \frac{\partial C}{\partial S}\mu S\,dt + \frac{\partial C}{\partial S}\sigma S\,dW$$

$$\boxed{= \left( -\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt}$$

$$\Rightarrow \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC \quad \text{Black-Scholes Equation}$$

Notes:

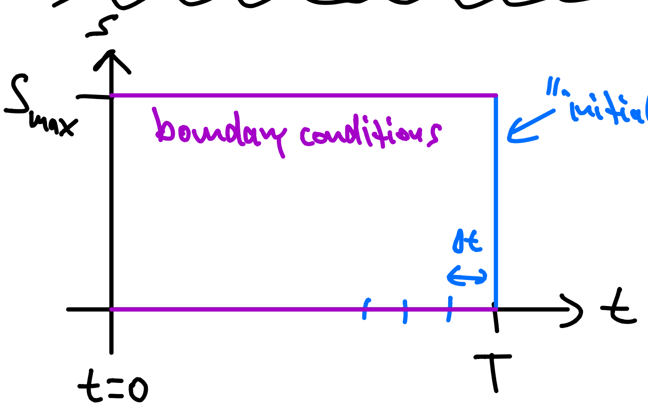
- independent of $\mu$ (option price only depends on volatility!)
- second order PDE
- by a change of variables this eq. can be transformed into a heat equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \quad (\text{or } c\,\Delta u \text{ for higher dim.})$$

- backward drift-diffusion equation (specify $C(S,T)$, solve for $C(S,0)$)

We specify $C(S,T) = \text{payoff} \overset{\text{European calls}}{=} \max(0, S-K)$ ← strike price

## 4.2 Discrete Finite Differences




"initial" condition $C(S,T) = \text{payoff}$

boundary conditions:

$C(t,0) = 0$

$C(t, S_{max}) = S_{max} - Ke^{-r(T-t)}$

(actually a sol. to BS eq.)

or $C(t, S_{max}) = S_{max}$ (since $S_{max}$ should be choosen much bigger than $K$)

partition $[0,T]$ into $M$ steps of size $\Delta t = \frac{T}{M}$, $t_j = j\Delta t$

partition $[0, S_{max}]$ into $N$ steps of size $\Delta S = \frac{S_{max}}{N}$, $S_i = i\Delta S$

call $C(t_j, S_i) = C_i^j$

$$\frac{\partial C}{\partial t} \approx \frac{C_i^{j+1} - C_i^j}{\Delta t} + \mathcal{O}(\Delta t)$$

one could choose $\frac{\partial C}{\partial S} \approx \frac{C_{i+1}^j - C_i^j}{\Delta S} + \mathcal{O}(\Delta S)$

alternatively (fix $j$):

(1) $C(S_i + \Delta S) = C(S_i) + \frac{\partial C}{\partial S}(S_i)\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_i)\Delta S^2 + \frac{1}{3!}\frac{\partial^3 C}{\partial S^3}\Delta S^3 + \mathcal{O}(\Delta S^4)$

(2) $C(S_i - \Delta S) = C(S_i) - \frac{\partial C}{\partial S}(S_i)\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S_i)\Delta S^2 - \frac{1}{3!}\frac{\partial^3 C}{\partial S^3}\Delta S^3 + \mathcal{O}(\Delta S^4)$

(1)−(2) $\Rightarrow C(S_i + \Delta S) - C(S_i - \Delta S) = 2\frac{\partial C}{\partial S}(S_i)\Delta S + \mathcal{O}(\Delta S^3)$

"centralized" derivative $\frac{\partial C}{\partial S} \approx \frac{C_{i+1}^j - C_{i-1}^j}{2\Delta S} + \mathcal{O}(\Delta S^2)$ (better error)

second derivative: (1)+(2)

$$\Rightarrow C(S_i + \Delta S) + C(S_i - \Delta S) = 2C(S_i) + \frac{\partial^2 C}{\partial S^2}(S_i)\Delta S^2 + \mathcal{O}(\Delta S^4)$$

$$\Rightarrow \frac{\partial^2 C}{\partial S^2} \approx \frac{C_{i+1}^j - 2C_i^j + C_{i-1}^j}{\Delta S^2} + \mathcal{O}(\Delta S^2)$$ (also $\Delta S^2$ error)

## 4.3 Stability of Time-stepping Methods

<u>Ex.:</u> $\frac{dy}{dt} = \lambda y$ , solution: $y(t) = y_0 e^{\lambda t}$

consider $\lambda < 0$ , say also $|\lambda| >> 1$.

Explicit Euler method: $\frac{y^{j+1} - y^j}{\Delta t} = \lambda y^j$ (r.h.s. evaluated at $j$)

$\Rightarrow y^{j+1} = \lambda y^j \Delta t + y^j = (1 + \lambda \Delta t) y^j$

$\Rightarrow y^j = (1 + \lambda \Delta t)^j y_0$

here: if $y^j \to \lim_{t \to \infty} y_0 e^{\lambda t} \underset{(\lambda<0)}{=} 0$ , then scheme is stable

$\Rightarrow$ we need $|1 + \lambda \Delta t| < 1$

$\Rightarrow 1 + \lambda \Delta t < 1$ and $-1 - \lambda \Delta t < 1$

$\Rightarrow \lambda < 0$ $\qquad\qquad \Rightarrow \Delta t < \frac{2}{-\lambda}$ condition on smallness of $\Delta t$ for scheme to be stable

Implicit Euler method: $\frac{y^{j+1} - y^j}{\Delta t} = \lambda y^{j+1}$ (r.h.s evaluated at $j+1$)

$\Rightarrow (1 - \lambda \Delta t) y^{j+1} = y^j \Rightarrow y^{j+1} = \left( \frac{1}{1 - \lambda \Delta t} \right) y^j$

$\Rightarrow y^j = \left( \frac{1}{1 - \lambda \Delta t} \right)^j y^0$

now stability condition is $\left|\frac{1}{1-\lambda\Delta t}\right| < 1$ ; this always holds here (since $\lambda < 0$)

↳ unconditionally stable

## 4.4 Application to Heat Equation

consider $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ , initial value $V(x,0)$ (want to know $V(x,T)$)

(for BS: backwards)

$$\frac{\partial V}{\partial t}(x_i,t_j) = \frac{V(x_i,t_{j+1}) - V(x_i,t_j)}{\Delta t} + O(\Delta t)$$

$$\frac{\partial^2 V}{\partial x^2}(x_i,t) = \frac{V(x_{i+1},t) - 2V(x_i,t) + V(x_{i-1},t)}{\Delta x^2} + O(\Delta x^2)$$

$t = t_j$ : explicit scheme

$t = t_{j+1}$ : implicit scheme

denote $V(x_i,t_j) = V_i^j$

<u>explicit:</u> $$\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{\Delta x^2}$$

$$\Rightarrow \quad V_i^{j+1} = \frac{\Delta t}{\Delta x^2} V_{i+1}^j + \left(1 - \frac{2\Delta t}{\Delta x^2}\right) V_i^j + \frac{\Delta t}{\Delta x^2} V_{i-1}^j$$

note: need $\frac{\Delta t}{\Delta x^2} < 1$ for stability

<u>implicit:</u> $$\frac{V_i^{j+1} - V_i^j}{\Delta t} = \frac{V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}}{\Delta x^2}$$

$$\Rightarrow \quad V_i^j = -\underbrace{\frac{\Delta t}{\Delta x^2}}_{a} V_{i+1}^{j+1} + \left(1+\frac{2\Delta t}{\Delta x^2}\right) V_i^{j+1} - \frac{\Delta t}{\Delta x^2} V_{i-1}^{j+1}$$

$$\underbrace{\begin{pmatrix} (1+2a) & -a & & 0 \\ -a & (1+2a) & -a & \\ & -a & \ddots & \ddots \\ 0 & & \ddots & -a \\ & & & (1+2a) \end{pmatrix}}_{n\times n \text{ matrix } A} \underbrace{\begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_n^{j+1} \end{pmatrix}}_{\text{vector } \vec{V}^{j+1}} = \underbrace{\begin{pmatrix} V_1^j \\ V_2^j \\ \vdots \\ V_n^j \end{pmatrix}}_{\text{vector } \vec{V}^j}$$

$\Rightarrow$ need to solve tridiagonal system of equations to get $\vec{V}^{j+1}$ from $\vec{V}^j$

what happens at the boundary?

$V_0^{j+1}$ and $V_{n+1}^{j+1}$ are given by fixed boundary conditions

we have $V_1^j = -a V_2^{j+1} + (1+2a) V_1^{j+1} - a V_0^{j+1}$

$$V_n^j = -a V_{n+1}^{j+1} + (1+2a) V_n^{j+1} - a V_{n-1}^{j+1}$$

so with boundary conditions the tridiagonal system is

fixed

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & (1+2a) & -a & \\ & -a & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} V_1^{j+1} \\ \vdots \\ V_n^{j+1} \end{pmatrix} = \begin{pmatrix} V_1^j + a V_0^{j+1} \\ V_2^j \\ \vdots \\ V_{n-1}^j \\ V_n + a V_{n+1}^{j+1} \end{pmatrix}$$

fixed