Session 25
Nov. 29, 2018

## 4.5 Connection between Black-Scholes eq. and Formula

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

European call: $C(S,T) = \max(S-K,0)$

bdry cond.: $C(0,t) = 0$

we do several changes of variables to reduce it to the heat eq.

$$C(S,t) = B(S,\tau)e^{-r\tau}K \qquad , \tau = T-t$$

$\Rightarrow$ in terms of $B$: $-\frac{\partial B}{\partial \tau} + rS\frac{\partial B}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 B}{\partial S^2} = 0$

$$B(0,\tau) = 0 \quad , \quad B(S,0) = \max\left(\frac{S}{K}-1, 0\right)$$

to remove first derivative: $D(x,\tau) = B(S,\tau)$ , $x = \frac{S}{K}e^{r\tau}$

$$\Rightarrow \quad -\frac{\partial D}{\partial \tau} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D}{\partial x^2} = 0$$

to remove $\sigma$: $H(x,u) = D(x,\tau)$ , $u = \sigma^2\tau$

$$\Rightarrow -\frac{\partial H}{\partial u} + \frac{1}{2}x^2\frac{\partial^2 H}{\partial x^2} = 0$$

to remove $x^2$-prefactor: $\Theta(z,u) = H(x,u)$ , $z = \frac{u}{2} + \ln x$

$\Rightarrow \boxed{-\frac{\partial \Theta}{\partial u} + \frac{1}{2}\frac{\partial^2 \Theta}{\partial z^2} = 0}$ heat equation , $\Theta(z,0) = \max(1-e^{-z},0)$

How to solve the heat eq.?

inverse Fourier transform: $\theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} \hat{\theta}(k,u)\,dk$

plug into eq:

$$\frac{1}{\sqrt{2\pi}} \int e^{-ikz} \frac{\partial \hat{\theta}(k,u)}{\partial u}\,dk = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} \left(\frac{1}{2}(-ik)^2\right) \hat{\theta}(k,u)\,dk$$

solve $\frac{\partial \hat{\theta}(k,u)}{\partial u} = -\frac{k^2}{2} \hat{\theta}(k,u)$

$$\Rightarrow \quad \hat{\theta}(k,u) = e^{-\frac{k^2}{2}u} \hat{\theta}(k,0)$$

$$\Rightarrow \theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} e^{-\frac{k^2}{2}u} \underbrace{\hat{\theta}(k,0)}\,dk$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{iky} \theta(y,0)\,dy$$

$$= \frac{1}{2\pi} \int \underbrace{\left[\int e^{-ik(z-y)} e^{-\frac{k^2}{2}u}\,dk\right]} \theta(y,0)\,dy$$

$$= \int dk\, e^{-\frac{u}{2}\left[k^2 + 2\frac{ik(z-y)}{u} + \left(\frac{i(z-y)}{u}\right)^2 - \left(\frac{i(z-y)}{u}\right)^2\right]}$$

$$= \int dk\, e^{-\frac{u}{2}\left(k + i\frac{z-y}{u}\right)^2} e^{-\frac{(z-y)^2}{2u}}$$

$$= e^{-\frac{(z-y)^2}{2u}} \underbrace{\int dk\, e^{-\frac{u}{2}k^2}}$$

$k = \ell\sqrt{\frac{2}{u}}$

$$= \sqrt{\frac{2}{u}} \int d\ell\, e^{-\ell^2}$$

$$= \sqrt{\frac{2\pi}{u}}$$

$$\Rightarrow \Theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int e^{-\frac{(z-y)^2}{2u}} \Theta(y,0)\, dy$$

$\Rightarrow$ with B-S initial cond. $\Theta(y,0) = \max(1-e^{-y}, 0)$ we get

$$\Theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-\frac{(z-y)^2}{2u}} (1-e^{-y})\, dy$$

substituting back our changes of variables we get B-S formula.