Jacobs University
Spring 2018

April 24, 2018

Analysis II

Homework 10

Due on May 1, 2018

Problem 1 [12 points]: The chain rule
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and let $f : U \to V$ be differentiable at $p \in U$ and $g : V \to \mathbb{R}^j$ be differentiable at $f(p)$. Prove that then the composition $F := g \circ f : U \to \mathbb{R}^j$ is differentiable at p with derivative $DF|_p = Dg|_{f(p)}\, Df|_p$.

Problem 2 [14 points]: Directional derivatives and total derivatives
Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $h(\theta+\pi) = -h(\theta)$ for all θ, hence $h(\theta + 2\pi) = h(\theta)$. Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ via

$$f(x, y) = r \cdot h(\theta) \ ,$$

where we write $(x, y) = (r \cos(\theta), r \sin(\theta))$.

(a) Show that f is differentiable everywhere in $\mathbb{R}^2$ except possibly at $(x, y) = (0, 0)$.

(b) Show that all directional derivatives of f exist at $(0, 0)$.

(c) Give necessary and sufficient conditions on h in order to f to be differentiable at $(0, 0)$.

(d) Specifially for $h(\theta) = \cos^3 \theta$, compute all directional derivatives at the origin.

Problem 3 [14 points]: Local maxima and the gradient
Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}$ is continuously differentiable. We say that f *has a local maximum at* $p \in U$ if p has a neighborhood V so that $f(p) \geq f(x)$ for all $x \in V$.

(a) Show that f can have a local maximum at p only if all partial derivatives vanish at p.

(b) Show that f can have a local maximum at p only if all directional derivatives vanish at p.

(c) Specifically for $f(x, y) = 1 - x^2(1-x)^2 - 2y^2$, find a necessary condition for local maxima. Can you decide whether all your candidate points are indeed local maxima?

(d) Do the same for $f(x, y) = (x^2 - y^2)$.

(e) We define the *gradient of* f as $\nabla f := (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. State a necessary condition for local maxima of f in terms of the gradient.

Problem 4 [8 points]: Length

Let $\vec{x} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function, and let $r : \mathbb{R} \to \mathbb{R}$ be the function $r(t) := \|\vec{x}(t)\| = \sqrt{(\vec{x})_1^2 + (\vec{x})_2^2 + (\vec{x})_3^2}$. Let $t_0 \in \mathbb{R}$. Show that if $r(t_0) \neq 0$ then r is differentiable at t_0 and

$$r'(t_0) = \frac{\vec{x}'(t_0) \cdot \vec{x}(t_0)}{r(t_0)}.$$

Bonus Problem

No bonus problem this time :(But 8 out of the 48 points count as bonus points :)