Jacobs University May 1, 2018
Spring 2018

Analysis II

Homework 11

Due on May 8, 2018

Problem 1 [8 points]: Cauchy-Riemann differential equations and harmonic functions

A function $u\colon \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if it is C^2 and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(a) Show that $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$ are both harmonic.

(b) More generally, prove that whenever $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are two C^2 functions that satisfy the Cauchy-Riemann differential equations (see Problem 3 from homework sheet 9), then u and v are both harmonic.

(c) [**4 bonus points**] Can you give conditions on when a harmonic function can have a local maximum?

Problem 2 [18 points]: Twice differentiable

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy(x^2-y^2)/(x^2+y^2)$ for $(x,y) \neq (0,0)$.

(a) Is f twice partially differentiable on $\mathbb{R}^2 \setminus (0,0)$, and are the second derivatives continuous?

(b) Show that with $f(0,0) = 0$, the function f is twice partially differentiable at $(0,0)$.

(c) Compute $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$ at $(x,y) = (0,0)$. Should that be surprising?

Problem 3 [6 points]: Second Order Taylor

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^{-y^2} - x^2(y+1)$.

(a) Prove that $f \in C^2$ and write down the Taylor expansion to second order.

(b) Does f have local extrema? Are these also global extrema?

Problem 4 [8 points]: The wave equation

The famous *wave equation* in one (space) dimension is the equation $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$ for a C^2-function $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$.

(a) Verify that $f(t,x) = a\cos(x - ct) + b\sin(x - ct)$ solves this wave equation for arbitrary $a, b \in \mathbb{R}$.

(b) Under which conditions on $g\colon \mathbb{R} \to \mathbb{R}$ is the function $f(t,x) = g(x - ct)$ a solution to the wave equation?

(c) The wave equation in n space dimensions is $\dfrac{\partial^2 f}{\partial t^2} = c^2 \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 f}{\partial x_i^2}$ for a C^2-function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ and $c \in \mathbb{R}$. Verify that for every $k \in \mathbb{R}^n$ (the wave direction vector) there is a solution $f(t,x) = \cos(\langle k, x\rangle - \omega t)$ for appropriate $\omega \in \mathbb{R}$, where $\langle k, x\rangle = \sum_{i=1}^{n} k_i x_i$ is the scalar product.

(d) [**4 bonus points**] Find more general solutions to the wave equation.