Jacobs University
Spring 2018

May 8, 2018

Analysis II

Homework 12

Due on May 15, 2018

Problem 1 [10 points]: Very basic vector calculus

Let us formally define the nabla operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Then the three classical vector differential operators *gradient*, *curl*, and *divergence* are defined as

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \quad \text{for } f : \mathbb{R}^3 \to \mathbb{R},$$

$$\text{curl } g = \nabla \times g = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}, \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}, \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right), \quad \text{for } g : \mathbb{R}^3 \to \mathbb{R}^3,$$

$$\text{div } h = \nabla h = \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z}, \quad \text{for } h : \mathbb{R}^3 \to \mathbb{R}^3.$$

(a) Show that every C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies curl grad $f = 0$.

(b) Show that every C^2 function $g : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies div curl $g = 0$.

(c) Show that every C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies div grad $f = \Delta f$, where $\Delta = \nabla\nabla$ is the *Laplace operator*, i.e.,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

(d) Show that every C^2 function $g : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies curl curl f = grad div $f - \Delta f$.

(e) Let $E, B : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be C^2 ($E = E(t, x)$, $B = B(t, x)$), and let them satisfy Maxwell's equations in vacuum

$$\text{div } E = 0, \quad \text{div } B = 0, \quad \text{curl } E = -\frac{\partial B}{\partial t}, \quad \text{curl } B = \frac{1}{c^2}\frac{\partial E}{\partial t},$$

where $c > 0$. Show that this implies that E and B satisfy a wave equation, i.e., that

$$\frac{\partial^2 E}{\partial t^2} = c^2 \Delta E, \quad \frac{\partial^2 B}{\partial t^2} = c^2 \Delta B.$$

Note that the wave equation is sometimes written as $\Box E = 0$, where $\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ is called the *d'Alembert operator*.

Problem 2 [12 points]: Maxima and minima

Let $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ be C^2, and let $\nabla f(p) = 0$ for some $p \in U$. Prove that if the Hessian $H_f(p)$ is positive definite (i.e., $h^T H_f(p) h > 0$ for all $0 \neq h \in \mathbb{R}^n$, then f has a local minimum at p. *Hint: Use the second order Taylor expansion from class. One strategy would be to consider h with $\|h\| = 1$ first (does $h \mapsto h^T H_f(p) h$ have a minimum on this set?). Then generalize by setting $\tilde{h} = th$ with $t \geq 0$.*

Problem 3 [10 points]: Two-dimensional polar coordinates

Consider the map $P \colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ given by $(r, \phi) \mapsto (r \cos\phi, r \sin\phi) =: (x, y)$.

(a) For which k is this map a C^k-function?

(b) Compute $DP|_{(r,\phi)}$ for arbitrary (r, ϕ). For which (r, ϕ) does P have a local inverse?

(c) Under which conditions is $P(r, \phi) = P(r', \phi')$? Show that P is a diffeomorphism from $\mathbb{R}^+ \times (0, 2\pi)$ to a certain subset $U \subset \mathbb{R}^2$; what is U? Specify an explicit inverse map $K := P^{-1}$ to P on this domain.

Problem 4 [8 points]: Three-dimensional polar coordinates

Consider the map $Q \colon \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ given by

$$Q(r, \theta, \phi) = (r \cos\theta \cos\phi, r \cos\theta \sin\phi, r \sin\theta) =: (x, y, z).$$

In analogy to the previous problem, discuss how often Q is differentiable, compute DQ, and show at which points (x, y, z) the map Q has a local inverse. Discuss injectivity of Q and find as large as possible a domain on which Q is invertible.

Bonus Problem 1 [8 points]: Newton's Method in Several Variables

Give a proof of the following theorem. Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}^n$ is C^1 with $f(\xi) = 0$. Define $N_f \colon U \to \mathbb{R}^n$ via $N_f(x) = x - (Df(x))^{-1} f(x)$. Then ξ has a neighborhood $U' \subset U$ with $\|N_f(x) - \xi\| < \frac{1}{2}\|x - \xi\|$.