Jacobs University
Spring 2018

February 19, 2018

Analysis II

Homework 2

Due on February 26, 2018

Problem 1 [15 points]: Properties of the Riemann-Stieltjes integral
Give careful proofs of the following results stated in class. The Riemann-Stieltjes integral has the properties:

(1) *Linearity in f:* If $f_1, f_2 \in \mathcal{R}(\alpha)[a,b]$, then $f_1 + f_2 \in \mathcal{R}(\alpha)[a,b]$ with

$$\int (f_1 + f_2)\, d\alpha = \int f_1\, d\alpha + \int f_2\, d\alpha,$$

and if $f \in \mathcal{R}(\alpha)[a,b]$ and $\lambda \in \mathbb{R}$, then $\lambda f \in \mathcal{R}(\alpha)[a,b]$ with

$$\int \lambda f\, d\alpha = \lambda \int f\, d\alpha.$$

In other words, $\int_a^b$ is an $\mathbb{R}$-linear map from $\mathcal{R}(\alpha)[a,b]$ to $\mathbb{R}$.

(2) *Monotonicity:* If $f_1, f_2 \in \mathcal{R}(\alpha)[a,b]$ and $f_1(x) \leq f_2(x)$ for all x, then $\int f_1\, d\alpha \leq \int f_2\, d\alpha$.

(3) *Additivity of Intervals:* If $f \in \mathcal{R}(\alpha)[a,b]$ and $c \in [a,b]$, then also $f \in \mathcal{R}(\alpha)[a,c]$ and $f \in \mathcal{R}(\alpha)[c,b]$ and

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$$

and conversely.

(4) *Standard Estimate of Integral:* If $f \in \mathcal{R}(\alpha)[a,b]$ and $f(x) \leq M$ for all $x \in [a,b]$, then $\int_a^b f\, d\alpha \leq M(\alpha(b) - \alpha(a))$ (and similarly for lower bounds).

(5) *Linearity in α:* If $f \in \mathcal{R}(\alpha_1)[a,b] \cap \mathcal{R}(\alpha_2)[a,b]$ and $\lambda_1, \lambda_2 \in \mathbb{R}_0^+$, then we have $f \in \mathcal{R}(\lambda_1\alpha_1 + \lambda_2\alpha_2)[a,b]$ with

$$\int_a^b f\, d(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda_1 \int_a^b f\, d\alpha_1 + \lambda_2 \int_a^b f\, d\alpha_2.$$

Problem 2 [8 points]: When the integral is zero
Let $\alpha : [a, b] \to \mathbb{R}$ be a monotonically increasing bounded function. Suppose f is a continuous and non-negative function on $[a, b]$ and $f \in \mathcal{R}(\alpha)[a, b]$. Is it true that if $\int_a^b f \, d\alpha = 0$ then $f(x) = 0$ for all $x \in [a, b]$? If not, give conditions on α for which this is true (with proof).

Problem 3 [8 points]: Simultaneous discontinuity of f and α
Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous except at one point $c \in (a, b)$, and let $\alpha\colon [a, b] \to \mathbb{R}$ be monotonically increasing and bounded. Suppose α is not continuous at c.

(a) Is it possible that $f \in \mathcal{R}(\alpha)[a, b]$? Give an example or prove there is none.

(b) Is it possible that $f \notin \mathcal{R}(\alpha)[a, b]$? Give an example or prove there is none.

(c) **Bonus Question [2 points]:** Try to find (and prove) necessary and sufficient conditions for $f \in \mathcal{R}(\alpha)[a, b]$.

Problem 4 [9 points]: Monotone functions are integrable
Let α be a monotonically increasing continuous function on $[a, b]$. Prove that if $f : [a, b] \to \mathbb{R}$ is monotone then $f \in R(\alpha)[a, b]$.

Bonus Problem 1 [3 points]: Sum of α's
Suppose $f \in \mathcal{R}(\alpha_1 + \alpha_2)$. Does this imply that $f \in \mathcal{R}(\alpha_1)$?

Bonus Problem 2 [3 points]: Integration of composition
Suppose $f\colon [a, b] \to [m, M]$ is continuous and $g\colon [m, M] \to \mathbb{R}$ is bounded and Riemann-integrable. Does this imply that $g \circ f$ is Riemann-integrable? (If there are no counterexamples here, then what if f is Riemann-integrable but not necessarily continuous?)