Jacobs University
Spring 2018

March 5, 2018

Analysis II

Homework 4

Due on March 12, 2018

Problem 1 [12 points]: Convergent and divergent series
Which of the following series converge absolutely, which ones conditionally, which ones diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{(2n-1)(2n+1)}}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{(3n-1)3n(3n+1)}}$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos x^n}{n^2}$, for $x \in \mathbb{R}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$;

(e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n!}$;

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^p n}$, depending on $p \in \mathbb{R}$;

(g) $\sum_{n=1}^{\infty} (-1)^n n^{-s}$, depending on $s \in \mathbb{R}$.

Problem 2 [8 points]: Taylor series of logarithm

(a) Prove that $\int_1^x \frac{dt}{t} = \ln x$ for every $x > 1$.

(b) Show that $\displaystyle\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ for $x \in (-1, 1)$. (*Hint:* $\frac{1}{1+x}$.)

(c) What can you say about the behavior of the power series of $\ln(1+x)$ at the end points (-1 and 1) of the interval? Conclude that $\displaystyle\ln 2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Problem 3 [6 points]: Dirichlet's test on convergence of series
Prove Dirichlet's test: Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of complex and real numbers respectively, so that

- the partial sums $A_N = \sum_{n=0}^{N} a_n$ form a bounded sequence;
- $b_0 \geq b_1 \geq b_2 \geq \cdots \geq 0$;
- $\lim_{n\to\infty} b_n = 0$.

Then the series $\sum_{n=0}^{\infty} a_n b_n$ converges. (*Hint: If A is a bound for all A_N, then what is a bound for the $b_n A_N$ for fixed n?*)

Problem 4 [14 points]: The Riemann zeta function
For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ define the Riemann zeta function via $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.

(a) Show that the series defining the ζ function converges for every real $s > 1$.

(b) Show that it converges absolutely for complex s with $\operatorname{Re} s > 1$.

(c) Show that it is continuous and that $\zeta(s) \to \infty$ as $s \searrow 1$.

(d) Show that $\zeta(s) = s\int_1^\infty \frac{\lfloor x \rfloor}{x^{s+1}}\,dx$ for $s > 1$, where $\lfloor x \rfloor$ is the floor function, i.e., $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (*Hint: compute $\sum_{n=1}^{N}(\ldots) - s\int_1^N(\ldots)$.*)

(e) **[2 bonus points]** Show that $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{(x - \lfloor x \rfloor)}{x^{s+1}}\,dx$ for $s > 1$.

(f) **[2 bonus points]** Conclude that there is a continuous function $f\colon [0,\infty) \to \mathbb{R}$ so that $\zeta(s) = \frac{1}{s-1} + f(s)$; in other words, the singularity of ζ at $s = 1$ can be removed by subtracting a simple pole. (*Note: In fact, $\zeta(s) - 1/(s-1)$ extends to a (complex) differentiable function on all of $\mathbb{C}$, and one of the most famous open questions in mathematics, the* Riemann hypothesis*, is whether there exists an $s \in \mathbb{C}$ with $1/2 < \operatorname{Re} s < 1$ such that $\zeta(s) = 0$.*)

Bonus Problem 1 [4 points]: Abel's test on convergence of series
Prove Abel's test: Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of complex and real numbers respectively, so that

- the series $\sum_{n=0}^{\infty} a_n$ converges;
- the sequence $\{b_n\}_{n\in\mathbb{N}}$ is monotone and bounded.

Then the series $\sum_{n=0}^{\infty} a_n b_n$ converges. (*Hint: One could prove Abel's test using Dirichlet's test.*)