Jacobs University — March 12, 2018
Spring 2018

Analysis II

Homework 5

Due on April 3, 2018

Problem 1 [8 points]: More about convergence: root and ratio test

(a) State carefully the root test and the ratio test for series: in which cases do they imply convergence (absolute or conditional?) or divergence, in which cases are they inconclusive?

(b) Give three examples that were not yet treated in class: one each where the root test proves convergence of a series, where it proves divergence, and where it is not conclusive.

(c) Same for the ratio test (different examples, please).

Problem 2 [8 points]: More about convergence
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Show that convergence of $\sum_n a_n$ implies convergence of $\displaystyle\sum_n \frac{\sqrt{a_n}}{n}$.

Problem 3 [16 points]: More about convergence: infinite products
Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers. Define $P_n = \prod_{i=0}^n p_i$. If the sequence $(P_n)_{n\in\mathbb{N}}$ converges then we denote its limit by $\prod_{n=0}^\infty p_n$. If the limit exists and is not zero, then we say that the *infinite product* $\prod_{n=0}^\infty p_n$ converges, otherwise the product is said to diverge.

(a) Assume that p_n are positive real numbers. Show that $\prod_{n=0}^\infty p_n$ converges if and only if $\sum_{n=0}^\infty \ln p_n$ converges.

(b) Show that if $p_n \geq 1$ for all n this latter condition is equivalent to the fact that $\sum_n (p_n - 1)$ converges.

(c) Find a sequence $(p_n)_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^\infty (p_n - 1)$ converges but $\prod_{n=1}^\infty p_n$ diverges.

(d) Find a sequence $(p_n)_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^\infty (p_n - 1)$ diverges but $\prod_{n=1}^\infty p_n$ converges (and is greater than zero).

(e) **[2 bonus points]** Use the integral $\int_0^{\pi/2} \cos^n x\, dx$ to prove *Wallis's formula* for π:

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{\pi}{2}.$$

Hint: Use the "squeeze theorem" (squeeze your sequence between two known ones).

(f) **[1 bonus point]** Show that

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right)$$

and obtain another proof of Wallis's formula.

Problem 4 [8 points]: More about convergence: Dirichlet series
A *Dirichlet series* is a series of the form $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where all $a_n \in \mathbb{C}$.

(a) Show that there is an $s_1 \in \mathbb{R} \cup \{\pm\infty\}$ so that the series converges absolutely for $s > s_1$, and even for all $s \in \mathbb{C}$ with $\operatorname{Re} s > s_1$, but not for $s < s_1$. (This s_1 is called the *absolute convergence abscissa.*)

(b) **[2 bonus points]** It may happen that the series converges conditionally for $s < s_1$. Show that this is possible only when $s \geq s_1 - 1$.

(c) **[2 bonus points]** Show that there is an $s_0 \in \mathbb{R} \cup \{\pm\infty\}$ so that the series converges conditionaly for $s > s_0$ and even for all $s \in \mathbb{C}$ with $\operatorname{Re} s > s_0$, but not for $s < s_0$. (This s_0 is called the *conditional convergence abscissa.*)

(d) **[1 bonus point]** What are the absolute and conditional convergence abscissas for $f(s) = \sum_{n\geq 1} (-1)^n n^{-s}$?

March 12, 2018

Analysis II

Homework 6

Due on April 3, 2018

Problem 1 [12 points]: Rectifiable graphs and their arc length
Let $I = [a, b] \subset \mathbb{R}$ be a closed and bounded interval and $f\colon I \to \mathbb{R}$ a continuous function. The *graph* of f is the curve $\Gamma\colon I \to \mathbb{R}^2$ given by $t \mapsto (t, f(t))$.

(a) If f is continuously differentiable, express the length of the graph of f in terms of an integral.

(b) Find the length of the graph of $f(x) = \sqrt{1 - x^2}$ for $x \in [0, 1]$. Why should you have expected this answer?

(c) Find the length of the α-helix $\gamma\colon [0, x] \to \mathbb{R}^3$ given by $\gamma(t) = (\cos t, \sin t, \alpha t)$, for $\alpha \in \mathbb{R}$.

(d) Find the length of the graph of the function $t \mapsto \cosh t = (e^t + e^{-t})/2$ over the interval $[0, x]$.

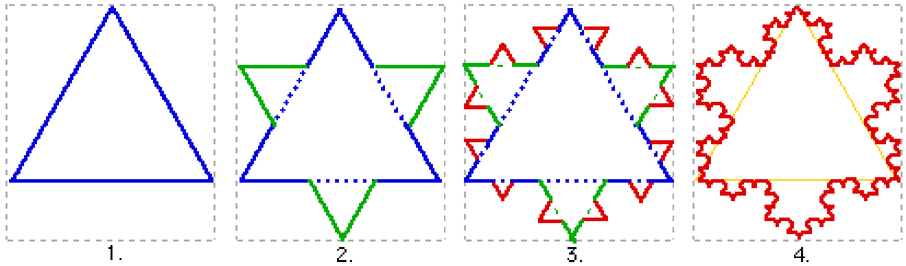
Problem 2 [12 points]: The snowflake: a non-rectifiable curve (the "von Koch"-curve)
Let us take a closer look at the von-Koch curve introduced in class. It is constructed iteratively in the following way:

1. Start with an equilateral triangle with sides of unit length.
2. On the middle third of each of the three sides, build an equilateral triangle with sides of length 1/3. Erase the base of each of the three new triangles.
3. On the middle third of each of the twelve sides, build an equilateral triangle with sides of length 1/9. Erase the base of each of the twelve new triangles.

The boundary after the n-th step is a piecewise linear curve, say γ_n; parametrize it as a map $\gamma_n\colon [0, 1] \to \mathbb{R}^2$ so that $\|\gamma_n'\|$ is constant.

(a) How many pieces does each γ_n have and what is their total length?

(b) Show that the sequence of maps γ_n converges uniformly to a continuous map $\gamma\colon [0, 1] \to \mathbb{R}^2$.



(c) Show that γ is not rectifiable.

(d) **[1 bonus point]** Show that the image of the snowflake curve is homeomorphic to the unit circle. (More precisely, find a continuous injective and surjective map from the circle to the image of the snowflake curve; continuity of the inverse will follow. How?)

(e) **[1 bonus point]** Prove that the snowflake is nowhere differentiable.

Problem 3 [16 points]: The hyperbolic metric

On the upper half plane $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \colon y > 0\}$, define the *hyperbolic length* of a continuously differentiable curve $\gamma \colon [a,b] \to \mathbb{H}$ (with $\gamma(t) = (x(t), y(t))$) by

$$\Lambda_H(\gamma) := \int_a^b \frac{\|\gamma'(t)\|}{y(t)}\, dt\,.$$

This is (up to a multiple) the only metric on $\mathbb{H}$ that is preserved by all Möbius transformations $z \mapsto (az+b)/cz+b$ with $a,b,c,d \in \mathbb{R}$ and $ad - bc = 1$, where $z = x + iy = (x,y) \in \mathbb{H}$; the purpose of this problem is to prove this invariance.

(a) Compute the length of the curves $\gamma_n \colon [1,2] \to \mathbb{H}$ given by $\gamma_n(t) = (0, 2^n t)$ connecting the two points $(0, 2^n)$ and $(0, 2^{n+1})$.

(b) Define a mapping $T_u \colon \mathbb{H} \to \mathbb{H}$ given by $T_u(x,y) := (x+u, y)$ (with $u \in \mathbb{R}$) (horizontal translation). Show that any two curves γ and $T_u \circ \gamma$ have equal lengths (if defined).

(c) Do the same for the mapping $S_r \colon \mathbb{H} \to \mathbb{H}$ given by $S_r(x,y) = (rx, ry)$, where $r > 0$ (scaling by a factor r).

(d) Do the same for the mapping $I \colon \mathbb{H} \to \mathbb{H}$ given by $I(x,y) = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$ (inversion in the unit circle).

(e) Conclude that the same result holds for any map $M(z) = \frac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad - bc = 1$ (where $z = x + iy$). (*Hint: this does not involve a lot of writing.*)

Bonus Problem 1 [2 points]: Devil's staircase

Find the length of the graph of the Devil's Staircase function (see Bonus Problem 2 from Homework 1).

Bonus Problem 2 [4 points]: Space-filling curves
The goal of this problem is to prove that there is a map $\phi\colon [0,1] \to [0,1]^2$ (from the closed unit interval to the closed unit square) which satisfies any two of the three properties: *injectivity*, *surjectivity* and *continuity*. (If ϕ satisfied all three properties, it would be a continuous bijection, but then $[0,1]$ and $[0,1]^2$ would be homeomorphic, which is not the case. Why?)

(a) Find an injective continuous map $\phi\colon [0,1] \to [0,1]^2$.

(b) Find an injective surjective map $\phi\colon [0,1] \to [0,1]^2$.

Hint: Write $x \in [0,1]$ in base 2 as $x = \sum_{k\geq 1} b_k 2^{-k}$, where $b_i \in \{0,1\}$ are binary digits. Remember that this representation has ambiguities!

(c) Find a continuous surjective map $\phi\colon [0,1] \to [0,1]^2$ (i.e. a *space-filling curve*).

Hint: Construct a sequence of continuous maps $\phi_n\colon [0,1] \to [0,1]^2$ that converges uniformly to a limiting map, so that for every $(x,y) \in [0,1]^2$ there is a $t \in [0,1]$ such that $\|(x,y) - \phi_n(t)\| < 2^{-n/2}$. This can be done by considering continuous curves going through the vertices of some equally-spaced grids in $[0,1]^2$.