Jacobs University — April 3, 2018
Spring 2018

Analysis II

Homework 7

Due on April 10, 2018

Problem 1 [20 points]: Picard-Lindelöf

Suppose that $v\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field that satisfies a Lipschitz condition with Lipschitz constant $L > 0$ and set $\varepsilon := \frac{1}{2L}$. For $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, define

$$X := \{\gamma\colon [t_0, t_0 + \varepsilon] \to \mathbb{R}^n, \gamma \text{ is continuous}, \gamma(t_0) = x_0\}$$

endowed with the supremum metric $d(\gamma_1, \gamma_2) := \sup_{t\in[t_0,t_0+\varepsilon]} \|\gamma_1(t) - \gamma_2(t)\|$. Finally, define the Picard-Lindelöf operator $P\colon X \to X$ by $P_\gamma(s) = \int_{t_0}^{s} v(\gamma(t))\, dt + x_0$. *Note: Some of the exercises were already partly done in class. Writing down a nice and complete proof possibly repeating some steps from class in the exercise here.*

(a) Show that indeed for every $\gamma \in X$ also $P_\gamma \in X$. Prove that X is complete.

(b) Show that P_γ is a contraction, i.e., $d(P_{\gamma_1}, P_{\gamma_2}) \leq \frac{1}{2} d(\gamma_1, \gamma_2)$ here. Conclude that P_γ has a unique fixed point (you may use the Banach Fixed Point Theorem here without proof.)

(c) Show that every curve in $P(X)$ is C^1. Is P surjective?

(d) Show that the vector field v has a unique solution curve $\gamma\colon [t_0, t_0 + \varepsilon] \to \mathbb{R}^n$ with $\gamma(t_0) = x_0$.

(e) Show that the vector field has a unique solution curve $\gamma\colon [t_0, \infty) \to \mathbb{R}^n$ with $\gamma(t_0) = x_0$.

Problem 2 [8 points]: The exponential differential equation

We consider the particularly simple vector field on $\mathbb{R}$ given by $v(x) = \lambda x$, with $\lambda \in \mathbb{R}$.

(a) Does v satisfy a Lipschitz condition? If so, what is the Lipschitz constant?

(b) Verify that $f(t) = ae^{\lambda t}$ is a solution curve, for every $a \in \mathbb{R}$.

(c) To show that $f(t) = ae^{\lambda t}$ is the unique solution with $f(0) = a$, show that an arbitrary solution curve $g(t)$ can always be written as $g(t) = ae^{\lambda t} \cdot h(t)$ for an appropriate curve $h\colon \mathbb{R} \to \mathbb{R}$, and so that h is C^1 if g is. Show that necessarily $h'(t) = 0$ and that h must be constant.

(d) **[3 bonus points]** Do the same discussion for $v(x, y) = (\lambda x, \mu y)$ on $\mathbb{R}^2$, with $\lambda, \mu \in \mathbb{R}$.

Problem 3 [12 points]: Separation of variables

Prove the following theorem that was proven in class. Let I, J be open intervals and $f : I \to \mathbb{R}$ continuous and $g : J \to \mathbb{R}$ continuous with $g(x) \neq 0$ for all $x \in J$. Then for all initial conditions $(t_0, x_0) \in I \times J$ there is an open interval $I_0 \subset I$ containing t_0 and a C^1 function $\gamma : I_0 \to \mathbb{R}$ with $\gamma(t_0) = x_0$ which solves the differential equation $\gamma'(t) = f(t)g(\gamma(t))$.
Hint: Proceed by first separating the variables formally and integrate both sides. (Is that possible? What properties do the integrals have?) Then you will need to take an inverse.

Bonus Problem 1 [5 points]: Exponential separation for Lipschitz vector fields

Suppose that $v\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field that satisfies a Lipschitz condition with Lipschitz constant $L > 0$. Show that if $\gamma_{1,2}\colon \mathbb{R} \to \mathbb{R}$ are two solution curves with $\|\gamma_1(t_0) - \gamma_2(t_0)\| \leq \delta$, then for all $t \geq t_0$ we have $\|\gamma_1(t) - \gamma_2(t)\| \leq \delta e^{L(t-t_0)}$. Discuss how sharp that bound is.
Hint: This is not so hard. Start by proving the Gronwall lemma.