Jacobs University
Spring 2018

April 10, 2018

Analysis II

Homework 8

Due on April 17, 2018

Note: This homework sheet is not so hard once you get used to compactness.

Problem 1 [5 points]: Subspace topology

Let (X, τ_X) be a topological space and $Y \subset X$. Prove that

$$\tau_Y = \{U_i \cap Y : U_i \in \tau_X\}$$

is indeed a topology on Y. *Note: This is trivial, but to get familiar with topologies carefully write down a nice formal proof.*

Problem 2 [10 points]: Continuity and compactness

Prove that the continuous image of any compact set is compact.

Problem 3 [25 points]: Some compactness lemmas

Prove the following lemmas about compactness:

(a) Every closed subset of a compact topological space is compact.

(b) In a Hausdorff space, every compact set is closed.

(c) Every continuous bijective map from a compact set to a Hausdorff space has a continuous inverse (i.e., is a homeomorphism).

Bonus Problem 1 [8 points]: Connectedness and path connectedness

A topological space (X, τ_X) is called *path connected* if any $x, y \in X$ can be connected by a path, i.e., there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. In class we showed or will show that path connectedness implies connectedness. Here we consider an example showing that the converse is in general not true.

We consider the *topologist's sine curve*

$$y(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{, for } x \in (0,1] \\ 0 & \text{, for } x = 0 \end{cases}$$

and its graph

$$\Gamma = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x \in (0,1] \right\} \cup \{(0,0)\}.$$

As usual, the graph $\Gamma \subset \mathbb{R}^2$ has the topology induced by the standard topology on $\mathbb{R}^2$. Prove that Γ is connected, but that it is not path connected.