Jacobs University — April 17, 2018
Spring 2018

Analysis II

Homework 9

Due on April 24, 2018

Problem 1 [10 points]: Connectedness

Prove that every path connected topological space is connected.

Hint: Recall the definitions of path connectedness and connectedness. One strategy is to assume that the space is path connected but not connected, and derive a contradiction.

Problem 2 [10 points]: Uniform continuity and compactness

Let $(X, d_X), (Y, d_Y)$ be metric spaces (and thus also topological spaces with the standard topology). A map $f : X \to Y$ is called uniformly continuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$ for all $x, x' \in X$. Prove that if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

Hint: By continuity, there is a $\delta(x)$ for each $x \in X$. Rephrase continuity in term of balls with radius $\delta(x)$ and then carefully use what compactness means.

Problem 3 [10 points]: Partial derivatives

Consider the function $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^3 - 3xy^2, 3x^2y - y^3)$ (i.e., $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$).

(a) Prove that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. (These will play a prominent role in complex analysis as the *Cauchy-Riemann differential equations.*)

(b) Show that, identifying complex numbers as $z = x+iy$, then simply $f(z) = z^3$. (This really is a special case of the general fact that complex-differentiable functions, in particular polynomials, satisfy the Cauchy-Riemann differential equations; there is also a converse.)

(c) Verify the Cauchy-Riemann differential equations for the complex exponential function.

Problem 4 [10 points]: Derivatives, partial derivatives, and continuity

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \frac{xy^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$.

(a) We consider first $(x, y) \neq (0, 0)$. Is f totally differentiable there? Do the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist? Do all directional derivatives exist? If so, compute all these derivatives for $(x, y) \neq (0, 0)$.

(b) Is there a value for $f(0, 0)$ so that the extension $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on all of $\mathbb{R}^2$?

(c) In case such an extension exists, do all directional derivatives at $(0,0)$ exist? Is f (totally) differentiable at $(0,0)$?

(d) Answer questions $(a), (b)$ and (c) also for $g(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

Bonus Problem 1 [8 points]: Banach fixed-point theorem

We have used the Banach fixed-point theorem (contraction mapping principle) already in the proof of the Picard-Lindelöf theorem and we are going to use it again in the proof of the inverse function theorem. So it's time to prove it.

In any metric space (X,d), a map $f : X \to X$ is called a *contraction* if there is an $r < 1$ such that $d(f(x), f(y)) \leq rd(x,y)$ for all $x, y \in X$. Prove that when (X,d) is a complete metric space, then any contraction $f : X \to X$ has a unique fixed point (i.e., a unique $x^* \in X$ with $f(x^*) = x^*$).

Hint: Uniqueness is easy. The fixed point can be constructed by defining a sequence $x_{n+1} = f(x_n)$*. Is this a Cauchy sequence?*