

Foundations of Mathematical Physics

Homework 10

Due on April 25, 2018

Problem 1 [6 points]: Laplace operator on the half line

- (a) We consider the operator $(-\Delta, C_0^\infty((0, \infty)))$ with $\Delta = \frac{d^2}{dx^2}$ as an unbounded operator on the Hilbert space $L^2([0, \infty))$. Is this operator self-adjoint? Is it essentially self-adjoint? Do self-adjoint extensions exist? Prove your answers using theorems from class. What is the adjoint operator?
- (b) We define

$$H_D^2(0, \infty) := \{\psi \in H^2(0, \infty) : \psi(0) = 0\},$$
$$H_N^2(0, \infty) := \{\psi \in H^2(0, \infty) : \psi'(0) = 0\},$$

and call $-\Delta_D = (-\Delta, H_D^2(0, \infty))$ the Dirichlet Laplacian and $-\Delta_N = (-\Delta, H_N^2(0, \infty))$ the Neumann Laplacian. Show that these operators are self-adjoint. Are they self-adjoint extensions of the operator from (a)? How do the unitary groups they generate differ? (Consider the reflexion of a wave packet at the origin. How does one get solutions of the Schrödinger equation with the specified boundary conditions from solution on \mathbb{R} ?)

- (c) Try to specify all self-adjoint extensions of the operator from (a) (using intuition from physics, if necessary).

Problem 2 [6 points]: δ -interaction in \mathbb{R}^3

On the Hilbert space $L^2(\mathbb{R}^3)$, we consider the operator $H_0 = -\Delta$ with domain $D(H_0) = C_0^\infty(\mathbb{R}^3 \setminus \{0\})$.

- (a) Do the equations $-\Delta\varphi_\pm = \pm i\varphi_\pm$ have solutions in the sense of distributions?
- (b) We now consider the equation from (a) as an ordinary differential equation and look for spherically symmetric solutions, i.e., for solutions $\varphi_\pm(r)$ of the equations

$$-\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}\right) \varphi_\pm(r) = \pm i\varphi_\pm(r).$$

Find all solutions to these equations. Do they have solutions in $L^2(\mathbb{R}^3)$?

- (c) Are the solutions from (b) also solutions to $H_0^* \varphi_{\pm} = \pm i \varphi_{\pm}$? (*Hint: Check whether $\langle \varphi_{\pm}, H_0 \psi \rangle = \langle \pm i \varphi_{\pm}, \psi \rangle$ for all $\psi \in D(H_0)$.*)
- (d) Find the deficiency spaces and deficiency indices. Does $(H_0, D(H_0))$ have self-adjoint extensions?
- (e) Is $(H_0, H^2(\mathbb{R}^3))$ a self-adjoint extension? The Cayley transform would allow us to find all self-adjoint extensions, but since we have not studied it carefully, we omit this exercise here. Can you find other self-adjoint extensions anyway (for extra points)?

Problem 3 [4 points]: Von Neumann theorem

Let $(H, D(H))$ be a densely defined symmetric linear operator. Prove that if there is a conjugation C with $CD(H) \subset D(H)$ and $CH\varphi = HC\varphi$ for all $\varphi \in D(H)$, then $N^+ = N^-$, i.e., the deficiency indices of H coincide.

Problem 4 [4 points]: Operator bounds

Let A, B be densely defined linear operators with $D(A) \subset D(B)$. Prove that the following two statements are equivalent:

- (i) There are $a, b \geq 0$ such that

$$\|B\varphi\| \leq a \|A\varphi\| + b \|\varphi\|$$

for all $\varphi \in D(A)$.

- (ii) There are $\tilde{a}, \tilde{b} \geq 0$ such that

$$\|B\varphi\|^2 \leq \tilde{a}^2 \|A\varphi\|^2 + \tilde{b} \|\varphi\|^2$$

for all $\varphi \in D(A)$.

Prove also that the infimum over all permissible a in (i) coincides with the infimum over all permissible \tilde{a} in (ii).