Jacobs University — May 2, 2018
Spring 2018

Foundations of Mathematical Physics

Homework 12

Due on May 9, 2018

Problem 1 [7 points]: Spectrum of multiplication operators
Let (X,μ) be a measure space and $f : X \to \mathbb{R}$ measurable. We define

$$D := \big\{\psi \in L^2(X,\mu) : f\psi \in L^2(X,\mu)\big\}.$$

(a) Prove that D is dense in $L^2(X,\mu)$ and that the multiplication operator (M_f, D) is self-adjoint on $L^2(X,\mu)$. *Hint: Approximate $\psi \in L^2(X,\mu)$, e.g., by multiplying it with the characteristic function of $f^{-1}([-n,n])$.*

(b) Prove that $\sigma(M_f) = \text{ess ran} f$, where $\sigma(M_f)$ is the spectrum of M_f, and

$$\text{ess ran} f := \big\{\lambda \in \mathbb{R} : \mu\big(f^{-1}(\lambda-\varepsilon,\lambda+\varepsilon)\big) > 0 \text{ for all } \varepsilon > 0\big\}$$

is the essential range. *Hint: The resolvent of a multiplication operator is useful here.*

Problem 2 [7 points]: Resolvent convergence
Let $\mathcal{H}$ be a Hilbert space, $(A, D(A))$ a self-adjoint operator, and $(A_n, D(A_n))$ a sequence of self-adjoint operators.

(a) Let $\sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{L}(\mathcal{H})} \le C$ and $A \in \mathcal{L}(\mathcal{H})$. Prove that then

$$\lim_{n\to\infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H})} = 0 \iff \exists z \in \mathbb{C}\setminus\mathbb{R} : \lim_{n\to\infty} \|R_z(A_n) - R_z(A)\|_{\mathcal{L}(\mathcal{H})} = 0.$$

Hint: Recall the estimate for the norm of the resolvent away from the real line. You can also use the second resolvent formula $R_z(A) - R_z(B) = R_z(A)(B-A)R_z(B)$.

(b) Prove that the following statements are equivalent:

(i) For every $\psi \in D(A)$ there is a sequence $(\psi_n)_n$ with $\psi_n \in D(A_n)$ and

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{\mathcal{H}} = 0 = \lim_{n\to\infty} \|A_n\psi_n - A\psi\|_{\mathcal{H}}.$$

(ii) There is a $z \in \mathbb{C}\setminus\mathbb{R}$ such that for all $\varphi \in \mathcal{H}$

$$\lim_{n\to\infty} \|R_z(A_n)\varphi - R_z(A)\varphi\|_{\mathcal{H}} = 0.$$

Problem 3 [6 points]: Example of functional calculus

We first give the definition of what a functional calculus is. Let $(H, D(H))$ be self-adjoint on a Hilbert space $\mathcal{H}$. A map that assigns an operator $f(H) \in \mathcal{L}(\mathcal{H})$ to each element $f : \mathbb{R} \to \mathbb{C}$ of a subalgebra $\mathcal{E}$ of the bounded Borel functions $\mathcal{B}(\mathbb{R})$, is called a functional calculus if:

(i) $f \mapsto f(H)$ is a homomorphism, i.e.,

$$(f + \alpha g)(H) = f(H) + \alpha g(H) \quad \text{and} \quad f(H)g(H) = (fg)(H) \quad \text{for all } f, g \in \mathcal{E}.$$

(ii) $f(H)^* = \bar{f}(H)$.

(iii) $\|f(H)\| \leq \|f\|_\infty$.

(iv) For $z \in \mathbb{C} \setminus \mathbb{R}$ and $r_z(x) = (x - z)^{-1}$, we have $r_z(H) = (H - z)^{-1}$ (resolvent).

(v) If $f \in C_0^\infty(\mathbb{R})$ vanishes on the spectrum of H, i.e., $\mathrm{supp} f \cap \sigma(H) = \emptyset$, then $f(H) = 0$.

Now let $\mathcal{H}$ be separable, $(H, D(H))$ a self-adjoint operator, and let (φ_n) be an orthonormal basis of eigenfunctions, i.e., $\varphi_n \in D(H)$ and there are (λ_n) with $H\varphi_n = \lambda_n \varphi_n$. For bounded and measurable $f : \mathbb{R} \to \mathbb{C}$ we define $f(H)$ by

$$f(H)\psi := \sum_{n=1}^{\infty} f(\lambda_n) \langle \varphi_n, \psi \rangle \, \varphi_n.$$

Prove that this defines a functional calculus.