Jacobs University February 28, 2018
Spring 2018

Foundations of Mathematical Physics

Homework 4

Due on March 7, 2018

Problem 1 [8 points]: Heat Equation

(a) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Determine the solution to the heat equation

$$\partial_t \psi(t,x) = \Delta_x \psi(t,x) \qquad \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}^d,$$
$$\psi(0,x) = \psi_0(x) \qquad \text{for all } x \in \mathbb{R}^d$$

by using the Fourier transform. Write the solution as

$$\psi(t,x) = \int_{\mathbb{R}^d} K(t, x-y)\psi_0(y)dy, \tag{1}$$

and explicitly state what the function $K : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is.

(b) Let $\psi_o \in C(\mathbb{R}^d)$ be bounded. Show that Equation (1) defines a bounded function $\psi \in C^\infty((0,\infty) \times \mathbb{R}^d)$ which solves the heat equation on $(0,\infty) \times \mathbb{R}^d$. Show also that ψ can be continuously extended by ψ_0 at $t = 0$, i.e., show that $\lim_{t\to 0} \psi(t,x) = \psi_0(x)$ for all $x \in \mathbb{R}^d$. (*Hint: Use Problem 4 from Homework 2.*)

Problem 2 [6 points]: Projectors
Let $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\| = 1$ and $N \geq 1$.

(a) For $1 \leq j \leq N$, let $p_j : L^2(\mathbb{R}^{dN}) \to L^2(\mathbb{R}^{dN})$ be defined by

$$(p_j\psi)(x_1,\ldots,x_N) := \varphi(x_j) \int \overline{\varphi(x_j)}\psi(x_1,\ldots,x_N)dx_j$$

for all $\psi \in L^2(\mathbb{R}^{dN})$ (here $\overline{z}$ denotes the complex conjugate of z). Show that p_j and $q_j = 1 - p_j$ are projectors, i.e., $p_j^2 = p_j$ and $q_j^2 = q_j$.

(b) Now let $P^{(N,k)} : L^2(\mathbb{R}^{dN}) \to L^2(\mathbb{R}^{dN})$ be defined by

$$P^{(N,k)} := \sum_{a\in\mathcal{A}_k} \prod_{j=1}^{N} (p_j)^{1-a_j}(q_j)^{a_j},$$

where $\mathcal{A}_k = \big\{a \in \{0,1\}^N : \sum_{j=1}^N a_j = k\big\}$. Show that $P^{(N,k)}$ is a projector for all $1 \leq k \leq N$, that $P^{(N,k)}P^{(N,\ell)} = 0$ for $\ell \neq k$, and that $\sum_{k=0}^N P^{(N,k)} = 1$ (where 1 is here the identity on $L^2(\mathbb{R}^{dN})$).

Problem 3 [6 points]: Gronwall's Lemma
Let us prove again a standard result from Analysis. Let $t \geq 0$ and let $\eta : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the estimate

$$\frac{d\eta(t)}{dt} \leq C(t)\Big(\eta(t) + \varepsilon\Big) \tag{2}$$

for some $\varepsilon \in \mathbb{R}$ and where $C : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that for all $t \geq 0$

$$\eta(t) \leq \exp\left(\int_0^t C(s)ds\right)\eta(0) + \left(\exp\left(\int_0^t C(s)ds\right) - 1\right)\varepsilon.$$