Jacobs University Spring 2018 — March 7, 2018

Foundations of Mathematical Physics

Homework 5

Due on March 14, 2018

Problem 1 [4 points]: Orthonormal basis
Prove that an orthonormal sequence (φ_j) in a Hilbert space is an orthonormal basis if and only if

$$\langle \varphi_j, \psi \rangle = 0 \quad \text{for all } j \in \mathbb{N} \quad \Rightarrow \quad \psi = 0.$$

Problem 2 [6 points]: Away from the support of $\widehat{\psi}$
Let $\psi \in \mathcal{S}$ and let its Fourier transform have compact support, i.e., $K = \operatorname{supp}(\widehat{\psi})$ is compact. Let U be an open ε-neighborhood of K, i.e., the distance between the complement of U and K is ε, i.e., $\operatorname{dist}(U^c, K) = \varepsilon > 0$. Prove that then for any $m \in \mathbb{N}$ there is a constant C_m such that for any t, x with $x/t \notin U$ and $|t| \geq 1$,

$$\left| \left(e^{-it(-\Delta)} \psi \right)(t, x) \right| \leq C_m \, (1 + |t|)^{-m}.$$

Hint: One could write the phase factor as $e^{i\alpha S}$ *with* $S(k) = \frac{kx - k^2 t/2}{1 + |t|}$ *and prove that*

$$e^{i\alpha S} = \left[\frac{1}{i\alpha} |\nabla_k S|^{-2} (\nabla_k S) \nabla_k \right]^m e^{i\alpha S},$$

and then integrate by parts.

Problem 3 [4 points]: Uncertainty
Prove the Heisenberg uncertainty principle. Let

$$\delta x_j := \langle \psi, \big(x_j - \langle \psi, x_j \psi \rangle \big)^2 \psi \rangle \qquad \delta p_j := \langle \psi, \big(p_j - \langle \psi, p_j \psi \rangle \big)^2 \psi \rangle$$

be the variances in the position and the asymptotic momentum distributions, where $p_j = -i\partial_{x_j}$ and $\langle f, g \rangle = \int \overline{f} g$. (Let's just take $\psi \in \mathcal{S}$ here.) Prove that

$$\delta x_j \delta p_j \geq \frac{1}{2}.$$

Problem 4 [4 points]: Refined Uncertainty
Prove the refined uncertainty principle (Hardy's inequality) on $\mathcal{S}(\mathbb{R}^3)$, i.e., that

$$\langle \psi, (-\Delta)\psi \rangle \geq \frac{1}{4} \langle \psi, |x|^{-2}\psi \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^3)$ (with the same notation as in the previous problem). *Hint: Look at the quantity* $\left[|x|^{-1}p_j|x|^{-1}, x_j\right]$, *where* $[A, B] := AB - BA$ *is the commutator. Note: This could be directly used to give a proof of the stability of hydrogen atoms.*

Problem 5 [2 points]: Projectors ctd.
We continue Problem 2 from Homework 4. Prove that

$$\langle \psi, q_1\psi \rangle = \langle \psi, \sum_{k=0}^{N} \frac{k}{N} P^{(N,k)}\psi \rangle$$

for all symmetric $\psi \in L^2(\mathbb{R}^{dN})$ (symmetric under exchange of variables).