

Foundations of Mathematical Physics

Homework 7

Due on April 4, 2018

Problem 1 [6 points]: Properties of generators

Prove the following properties of a generator H of a strongly continuous unitary group $U(t)$:

- (a) $U(t)D(H) = D(H)$ for all $t \in \mathbb{R}$, i.e., the domain $D(H)$ is invariant under $U(t)$,
- (b) $[H, U(t)]\psi := HU(t)\psi - U(t)H\psi = 0$ for all $\psi \in D(H)$,
- (c) H is symmetric, i.e., $\langle H\psi, \varphi \rangle = \langle \psi, H\varphi \rangle$ for all $\psi, \varphi \in D(H)$,
- (d) U is uniquely determined by H and H is uniquely determined by U .

Problem 2 [2 points]: Sobolev Lemma

Let $\ell \in \mathbb{N}_0$ and $f \in H^m(\mathbb{R}^d)$ with $m > \ell + \frac{d}{2}$. Prove that then $f \in C^\ell(\mathbb{R}^d)$ and $\partial^\alpha f \in C_\infty(\mathbb{R}^d)$ for all $|\alpha| \leq \ell$. *Hint: Riemann-Lebesgue and Cauchy-Schwarz.*

Problem 3 [6 points]: Group of translations

Let $a \in \mathbb{R}$ and $(T_a(t))_{t \in \mathbb{R}}$ the group of translations on $L^2(\mathbb{R})$, i.e., $(T_a(t)\psi)(x) := \psi(x - at)$.

- (a) Prove that the generator of T_a is $-ia \frac{d}{dx}$ with domain $H^1(\mathbb{R})$.
- (b) Prove that $T_a(t)$ converges weakly, but not strongly to 0 on $L^2(\mathbb{R})$ for $t \rightarrow \infty$.

Problem 4 [6 points]: Multiplication operators as generators

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable and let $M_V : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be the multiplication operator from Problem 4 in Homework sheet 3. Prove that M_V with domain $D(M_V) := \{\psi \in L^2(\mathbb{R}^d) : V\psi \in L^2(\mathbb{R}^d)\}$ generates a unitary group $U(t) = e^{-iVt}$ if $D(M_V)$ is dense in $L^2(\mathbb{R}^d)$. *Hint: The group property and strong continuity are easy, so the conditions from Def. 3.31 have to be checked. Note that the second condition from this definition needs to be checked only at $t = 0$ due to the group property and is easily established once the first condition is proven.*