# Foundations of Mathematical Physics

Summary Sheet

May 21, 2018

Here is a list of keywords and concepts from the different chapters that you should be able to define and explain for the exam:

### 1. Introduction

**Keywords:** Schrödinger equation of one and many particles; $|\psi|^2$ as a probability density; entanglement; wave functions of bosons and fermions.
**Concepts to explain:** What is the wave function and its dynamics and why are we interested in it.

### 2. The Free Schrödinger Equation

**Keywords:** Fourier transform; integrals with parameters; space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ with the semi-norms and metric; properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ (multiplication and differentiation, continuity, bijectivity, convolution); solution to the free Schrödinger equation (wave function as a map $\mathbb{R} \to \mathcal{S}(\mathbb{R}^d)$); pseudo-differential operators; tempered distributions (definition, examples, weak and weak* convergence, adjoints); differentiation, convolution and Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$; solution to free Schrödinger equation in $\mathcal{S}'(\mathbb{R}^d)$; long-time asymptotics of the free Schrödinger equation.
**Concepts to explain:** How do we extend the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ and all its properties to an operator on $\mathcal{S}'(\mathbb{R}^d)$? How and in which sense can we talk about solutions to the free Schrödinger equation? What is the long-time asymptotics of the free Schrödinger equation?

### 3. The Schrödinger Equation with Potential

**Keywords:** Banach and Hilbert spaces and related basic concepts and inequalities; bounded/continuous operators and their extensions; unitary operators; norm, strong, and weak convergence; strongly continuous unitary one-parameter groups and their generators; Sobolev space and Sobolev Lemma; the example of the translation operator; Riesz representation; Hilbert space adjoint; self-adjointness; bounded generators; unbounded operators; closedness and closeable operators; essential self-adjointness; criteria for (essential) self-adjointness; example of translation operators and Laplacian; deficiency spaces and indices; von Neumann theorem; resolvent and spectrum; relative and infinitesimal boundedness; Kato-Rellich.
**Concepts to explain:** How can we extend densely defined bounded operators? How can we understand the free propagator as a unitary group and with which properties? Why is symmetry not enough for unbounded operators to generate a unitary group? (What can go wrong?) What is self-adjointness and essential self-adjointness and why are we interested in it? What are the resolvent and the spectrum? How can we use Kato-Rellich to prove self-adjointness of relevant Hamiltonians?

**4. Spectral Theorem**

**Keywords:** three versions of the spectral theorem; Helffer-Sjöstrand formula; almost analytic extensions; functional calculus; multiplication operator version of the spectral theorem.

**Concepts to explain:** The three different versions of the spectral theorem and how they can be formulated in the finite dimensional case. How do we define a functional calculus and why do other alternatives not work for unbounded operators? What is a functional calculus? How does the multiplication operator version of the spectral theorem look like?

## List of Representative Problems from the Homework Sheets

Apart from the above definitions and explanations, problems similar to the following ones might be part of the exam:

- Sheet 1: Problems 2, 3
- Sheet 2: Problems 2, 3, 4
- Sheet 3: Problems 1, 2
- Sheet 4: Problem 1
- Sheet 5: Problems 3, 4
- Sheet 6: Problems 1, 3
- Sheet 7: Problems 1, 3 (b)
- Sheet 8: Problem 2
- Sheet 9: Problem 3
- Sheet 10: Problems 1, 2
- Sheet 11: Problems 1, 2, 3, 4
- Sheet 12: Problems 2, 3