Jacobs University
Fall 2019

September 21, 2019

Calculus on Manifolds

Homework 3

Due on October 2, 2019

Problem 1 [5 points]

Give a clean proof of the fact that for topological manifolds, connectedness implies path-connectedness.

Problem 2 [5 points]

Let M be a topological manifold and $p, q \in M$ with $p \neq q$.

(a) Show that there exists a continuous function $f : M \to [0, 1]$ with $f(p) = 0$ and $f(q) = 1$.

(b) Now suppose M is also a smooth manifold. Show that then there exists a smooth function $f : M \to [0, 1]$ with $f(p) = 0$ and $f(q) = 1$.

Problem 3 [3 points]

Let $\mathbb{P}^n$ be the n-dimensional real projective space. For distinct real $a_0, \ldots, a_n$, consider the function $f : \mathbb{P}^n \to \mathbb{R}$ defined by

$$f([x^0, \ldots, x^n]) = \frac{\sum_{k=0}^{n} a_k (x^k)^2}{\sum_{k=0}^{n} (x^k)^2}.$$

Show that f is indeed a well-defined function, and that it is smooth.

Problem 4 [3 points]

Show that the n-dimensional real projective space $\mathbb{P}^n$ (which we know is second-countable and Hausdorff) with the charts defined in class is a smooth manifold.

Problem 5 [4 points]

We consider the n-sphere $\mathbb{S}^n$, which, as a subspace of $\mathbb{R}^{n+1}$, we know is second-countable and Hausdorff.

(a) Let

$$U_i^+ = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i > 0\} \text{ and } U_i^- = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^i < 0\}.$$

Define homeomorphisms $\varphi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$ that orthogonally project on the $x^i = 0$ plane, thus proving that $\mathbb{S}^n$ is a topological manifold. Then show that $\{(U_i^\pm, \varphi_i^\pm)\}$ is a smooth atlas, and thus $\mathbb{S}^n$ a smooth manifold.

(b) Show that the stereographic projections that we defined in class are also a smooth atlas.

Bonus Problem [4 extra points]

Recall from Analysis that a set $A \subset \mathbb{R}^d$ has measure zero if for all $\varepsilon > 0$ there exists a countable family $\{B_k\}_k$ of balls in $\mathbb{R}^d$ such that

$$A \subset \bigcup_{k=1}^{\infty} B_k \quad \text{and} \quad \sum_{k=1}^{\infty} \mathrm{Vol}_d(B_k) < \varepsilon,$$

where Vol_d is the d-dimensional volume. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1. Show that for $m < n$, $f(\mathbb{R}^m)$ has measure zero. *Hint: Problem 3 (b) from Homework 1.*