Jacobs University
Fall 2019

November 7, 2019

Calculus on Manifolds

Homework 7

Due on November 14, 2019

Problem 1 [4 points]

Prove that every Lie group homomorphism has constant rank.

Problem 2 [4 points]

Let G^0 be the identity component of a Lie group G. Prove that G^0 is a Lie group, $\dim G^0 = \dim G$, and that G^0 is a normal subgroup of G.

Problem 3 [5 points]

We consider the group $O_n(\mathbb{R})$ of orthogonal real $n \times n$ matrices (i.e., matrices whose columns and rows are orthonormal).

(a) Prove that $O_n(\mathbb{R})$ is a compact group. *(Hint: Consider the map $g \mapsto g^T g$, where g^T is the transpose of g.)*

(b) Show that the image of the homomorphism $\det : O_n(\mathbb{R}) \to \mathbb{R}^*$ is the subgroup $\{-1, +1\}$.

(c) Now consider the special orthogonal group $SO_n(\mathbb{R})$, i.e., the subgroup of $O_n(\mathbb{R})$ consisting of matrices g with $\det g = 1$. Show that $SO_2(\mathbb{R})$ is isomorphic to the circle S^1, and thus abelian. Is $O_2(\mathbb{R})$ also abelian?

(d) Prove that $SO_3(\mathbb{R})$ is not abelian. *(Hint: Consider rotations with different axes.)*

Problem 4 [4 points]

Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The map f is called *proper* if for every compact set $K \subset Y$ the preimage $f^{-1}(K)$ is also compact.

(a) Give an example of a proper and an example of a non-proper continuous function $f : \mathbb{R} \to \mathbb{R}$.

(b) Suppose X is compact and Y is Hausdorff. Show that then $f : X \to Y$ is proper.

(c) Suppose $f : X \to Y$ is a bijective proper continuous map. Show that then f is a homeomorphism.

Problem 5 [3 points]

(a) Consider the standard embedding $S^1 \subset \mathbb{R}^2$. For each point $p = (x, y) \in S^1$, define $X(p) = (-y, x)$. Show that $X(p)$ is a basis for T_pS^1 for all $p \in S^1$.

(b) Consider the standard embedding $S^3 \subset \mathbb{R}^4$. For each point $p = (x, y, z, t) \in S^3$, define

$$\begin{aligned} X_1(p) &= (-y, x, t, -z) \\ X_2(p) &= (-z, -t, x, y) \\ X_3(p) &= (-t, z, -y, x). \end{aligned}$$

Show that $X_1(p), X_2(p), X_3(p)$ form a basis of T_pS^3 for all $p \in S^3$.