Jacobs University — November 21, 2019

Fall 2019

Calculus on Manifolds

Homework 9

Due on November 28, 2019

Problem 1 [**4 points**]

For any integer $n \geq 1$, by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ in the usual way, we can consider the odd-dimensional sphere $\mathbb{S}^{2n-1}$ as a subset of $\mathbb{C}^n$. Define a global flow on $\mathbb{S}^{2n-1}$ by $\theta(t, z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nowhere-vanishing vector field on $\mathbb{S}^{2n-1}$. For $n = 2$, find the integral curves of θ.

Problem 2 [**5 points**]

Let X and Y be two (smooth) vector fields on a smooth manifold M. Let θ_t be the local flow of Y. Prove that

$$\left.\frac{d}{dt}\right|_{t=0} \big((\theta_{-t})_* X\big) = [Y, X],$$

where $(\theta_{-t})_*$ is the pushforward of $\theta_{-t} : M \to M$, and $[X, Y]$ the Lie bracket. Start with the fact that for any smooth $f : M \to \mathbb{R}$ one can write

$$f(\theta_t(x)) = f(x) + t(Yf)(x) + t^2 E(x, t)$$

for some smooth $E(x, t)$ with $E(x, 0) = \frac{1}{2}(Y^2 f)(x)$, and then compute directly.

Problem 3 [**4 points**]

Consider the smooth manifold $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, and the smooth function $f : M \to \mathbb{R}, f(x, y) = \frac{x}{x^2+y^2}$. Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$, once in standard coordinates (x, y), and once in polar coordinates (r, φ).

Problem 4 [**3 points**]

Let M be a smooth manifold, and $\omega \in T_p^* M$. Suppose two local coordinates (x^j) and $(\tilde{x}^j)$ are given for some neighborhood of $p \in M$. How do the components of ω in one set of coordinates change when expressed in the other coordinates?

Problem 5 [**4 points**]

Let V be a vector space, then $V^* \otimes \ldots \otimes V^*$ (k times) is called the space of covariant k-tensors. A tensor T is called alternating if $T(\ldots, v_i, \ldots, v_j, \ldots) = -T(\ldots, v_j, \ldots, v_i, \ldots)$, and symmetric if $T(\ldots, v_i, \ldots, v_j, \ldots) = T(\ldots, v_j, \ldots, v_i, \ldots)$.

Now consider $V = \mathbb{R}^3$. Let (e^1, e^2, e^3) (note the upper indices) be the standard dual basis for $(\mathbb{R}^3)^*$.

(a) Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

(b) Can $e^1 \otimes e^2 + e^2 \otimes e^1 \in V^* \otimes V^*$ be written as $w^1 \otimes w^2$ for some $w^1, w^2 \in V^*$? (Prove your answer.)