Jacobs University November 28, 2019
Fall 2019

Calculus on Manifolds

Homework 10

Due on December 5, 2019

Problem 1 [1.5 points]

Let V be a finite dimensional vector space and $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ the alternation map. Let $\beta \in T^3(V^*)$. Explicitly compute $\mathrm{Alt}\beta(v_1, v_2, v_3)$ for any $v_1, v_2, v_3 \in V$.

Problem 2 [3 points]

(a) Let $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^\ell(V^*)$. Prove that $\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega$.

(b) Let $\omega^1, \ldots, \omega^k \in \Lambda^1(V^*)$ and $v_1, \ldots, v_k \in V$. Prove that

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det \omega^j(v_i).$$

Problem 3 [5 points]

Let $F : M \to N$ be a smooth map between smooth manifolds M, N, and let ω, η be differential forms on N, then the pullbacks $F^*\omega$ and $F^*\eta$ are differential forms on M.

(a) Prove that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(b) Prove that in any smooth chart,

$$F^*\left(\sum_I{}' \omega_I\, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F).$$

(c) Let $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^j))$ be overlapping smooth coordinate charts on the smooth n-manifold M. Prove that on $U \cap \tilde{U}$ we have

$$d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \ldots \wedge dx^n.$$

Problem 4 [6 points]

We consider the manifold $\mathbb{R}^n$. Recall that for a k-form ω on $\mathbb{R}^n$ we define the exterior derivative $d\omega$ as the $(k+1)$-form

$$d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J,$$

where $d\omega_J$ is the differential of $\omega_J : \mathbb{R}^n \to \mathbb{R}$. Prove that d has the following properties:

(a) d is $\mathbb{R}$-linear.

(b) For a smooth k-form ω and a smooth ℓ-form η we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$.

(d) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and ω a smooth k-form on $\mathbb{R}^m$, then

$$F^*(d\omega) = d(F^*\omega).$$

Problem 5 [3 points]
On $\mathbb{R}^3$, consider the 2-form

$$\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

(a) Compute Ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\theta).$$

(b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

Problem 6 [1.5 points]
Let $F : \mathbb{R}^2 \to \mathbb{R}^3, F(u, v) = (u^2, v^3, e^u - v)$ and let ω be the 2-form $\omega = xdx \wedge dy + zdx \wedge dz$ on $\mathbb{R}^3$. Compute the pullback $F^*\omega$.