Calculus on Manifolds

Session 1
Sep. 4, 2019

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Office: 112, Research I

Organization:

- see syllabus
- see website
- class: Wed., 8:15–9:30 (WH-4)
  Thu., 15:45–17:00 (EH-4)

- weekly homework
  ↳ see website
  ↳ due a week later before class (hand-in or mailbox entrance Res. I)
  ↳ 3 worst sheets do not count for grading
  ⇒ no late hand-ins, no excuses (except illness longer than a week)

- TA: Prabhat Devkota
  ↳ HW grading
  ↳ tutorial/office hour announced soon

- grade: 20% HW
  30% midterm (Thu., Oct. 24)
  50% final (tba)

note: if final exam grade better than midterm grade it replaces midterm grade

- book: "Lee – Introduction to Smooth Manifolds"

0. Overview / Motivation

(smooth) manifolds:

- generalizations of curves and surfaces
- things that "locally look like $\mathbb{R}^n$", but not necessarily globally (e.g., sphere, torus)
- but need not be embedded in some $\mathbb{R}^m$
- on basic level: topological manifolds
- more interesting: calculus (curvature, volume) $\Rightarrow$ need extra "smooth structure"
- differential geometry: more extra structure
  (e.g., inner product $\Rightarrow$ Riemannian, Lorentzian, symplectic manifolds)
  Riemannian $\to$ distance, angles; Lorentzian $\to$ spacetime; symplectic $\to$ phase space in classical mechanics

  $\hookrightarrow$ only briefly touched here, we rather provide general framework

- this class:
  - review of calculus in $\mathbb{R}^n$ (+ Linear algebra), topology
  - rigorous def. of smooth manifolds and smooth maps + all related aspects
    (tangent space, group actions, submanifolds, rank thm., embeddings, vector fields, tangent/cotangent/vector bundles)
  - differential forms
  - integration, Stokes thm.
  - Lie groups

1. Preparation

1.1 Review of Differentiation in $\mathbb{R}^n$

derivative = local approximation by linear map

recall: • vector $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \in \mathbb{R}^n$

• Scalar product $\langle x, y \rangle = \sum_{i=1}^{n} x^i \cdot y^i$

• norm: $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2}$

• Landau notation: a fct. $f: \mathbb{R}^m \to \mathbb{R}^n$ is

- $o(h)$ ("little o") if $\frac{\|f(h)\|}{\|h\|} \xrightarrow{h \to 0} 0$
- $O(h)$ ("big O") if $\limsup_{h \to 0} \frac{\|f(h)\|}{\|h\|} < \infty$

(note: sometimes $h \to a \in \mathbb{R}^m$ or $h \to \infty$ is used)

• linear map $L: \mathbb{R}^m \to \mathbb{R}^n$: $L(\lambda x + y) = \lambda L(x) + L(y)$, $\lambda \in \mathbb{R}$, $x, y \in \mathbb{R}^m$

• linear map + basis choice = matrix

↳ def. $e_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th position , $\{e_i\}$ = canonical basis of $\mathbb{R}^m$

↳ $L(e_1), \ldots, L(e_m)$ uniquely determine linear map
↓
column vectors of associated matrix

Def.: Let $U \subset \mathbb{R}^m$ be open. Then $f: U \to \mathbb{R}^n$ is called differentiable at $a \in U$ if there exists a linear map $L: \mathbb{R}^m \to \mathbb{R}^n$ s.t.

$$f(x+h) = f(x) + Lh + o(h) \quad \forall h \in \mathbb{R}^m$$

L = (total) derivative $= Df|_a = Df(a) = f'(a)$ (note: unique)

Ex.: • $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2$

$$\Rightarrow f(x+h) = \|x+h\|^2 = \|x\|^2 + 2\langle x, h\rangle + \|h\|^2$$

$$\Rightarrow Df(x)h = 2\langle x, h\rangle$$

in canonical basis: $Df(x) = 2(x^1, \dots, x^n)$

$$\Rightarrow Df(x)h = 2(x^1, \dots, x^n)\begin{pmatrix} h^1 \\ \vdots \\ h^n \end{pmatrix} = 2\langle x, h\rangle$$