Session 2
Sep. 5, 2019

Thm. (chain rule): Let $f: U \to V$ be differentiable at $x \in U$, $g: V \to W$ differentiable at $f(x) \in V$.
Then $g \circ f$ is differentiable at x and $D(g\circ f)(x) = Dg(f(x)) \circ Df(x)$.

Ex.: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2$, $g: \mathbb{R} \to \mathbb{R}$, $g(y) = e^y$ $\Rightarrow (g\circ f)(x) = g(f(x)) = e^{\|x\|^2}$

$\Rightarrow Df(x) = 2(x^1, \dots, x^n)$, $Dg(y) = e^y$ $\Rightarrow D(g\circ f)(x) = e^{\|x\|^2} 2(x^1, \dots, x^n)$
$(D(g\circ f)(x)h = 2e^{\|x\|^2}\langle x, h\rangle)$

Def. (partial derivative): For $f: \mathbb{R}^n \to \mathbb{R}^n$, we def. $\frac{\partial f}{\partial x^j} := \left(\frac{\partial f^1}{\partial x^j}, \dots, \frac{\partial f^n}{\partial x^j}\right)$

with $\frac{\partial f^i}{\partial x^j}(a) = \lim_{t\to 0} \frac{f^i(a+te^j) - f^i(a)}{t}$.

Matrix $\frac{\partial f^i}{\partial x^j}(a)$ called Jacobian matrix at a

Thm.: $f: U \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}^n$ differentiable at $a \in U \Rightarrow \frac{\partial f^i}{\partial x^j}$ exists for all j and $(Df(a))_{ij} = \sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j}(a)$

Converse?

Ex.: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

$\Rightarrow \frac{\partial f}{\partial x}((0,0)) = 0 = \frac{\partial f}{\partial y}((0,0))$
but f not continuous at $(0,0)$ (thus also not differentiable at $(0,0)$)

Def.: $f: U \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}^n$ with all partial derivatives continuous on $U \Rightarrow f \in C^1$ ("f is of class C^1")

Thm.: $f \in C^1 \Rightarrow f$ differentiable on U ($f \in C^1$ on $U \Leftrightarrow f$ continuously differentiable on U)

Def.: • $f \in C^k$: all (also mixed) partial derivatives of order k exist and are continuous

• $f \in C^0$: f cont.

• $f \in C^\infty$ or f smooth means $f \in C^k \; \forall k \geq 0$

• $\underset{\subset \mathbb{R}^n}{f: U} \to \underset{\subset \mathbb{R}^n}{V}$ diffeomorphism: smooth + smooth inverse (C^k-diffeomorphism: C^k + C^k inverse)

(U, V open)

Thm. (Schwarz): $f \in C^2 \Rightarrow \dfrac{\partial^2 f^i}{\partial x^j \partial x^k} = \dfrac{\partial^2 f^i}{\partial x^k \partial x^j}$ ($f \in C^k \Rightarrow$ all partial derivatives up to order k commute)

Def.: directional derivative of $f: \mathbb{R}^n \to \mathbb{R}$ in direction $v \in \mathbb{R}^n$ at $a \in \mathbb{R}$ is

$$D_v f(a) = \frac{d}{dt} f(a+tv)\Big|_{t=0}.$$

note: • $D_v f(a) \underset{\uparrow \text{chain rule}}{=} Df(a)v = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\, v^i = \langle \nabla f, v \rangle$

• linear $D_v(\lambda f + g)(a) = \lambda D_v f(a) + D_v g(a)$ ($\lambda \in \mathbb{R}$)

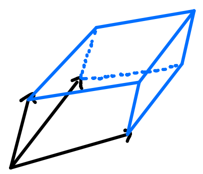
• product rule: $D_v(f \cdot g)(a) = (D_v f(a))\, g(a) + f(a)(D_v g(a))$

now: fundamental result

Thm. (Inverse Fct. Thm.): Let $f: U \to \mathbb{R}^n$ ($U \subset \mathbb{R}^n$ open) be C^k with $Df(a)$ invertible for some $a \in U$. Then $\exists V \subset U$ open, $a \in V$, s.t. $f|_V$ has inverse of class C^k and $W = f(V)$ is open. Moreover, $(Df^{-1})(f(x)) = (Df(x))^{-1} \; \forall x \in V$ (derivative of inverse = inverse of derivative).

note: if $Df(a)$ not invertible, then a is called critical point and $f(a)$ a critical value recall:

- matrix A invertible (or non-singular) $\Longleftrightarrow \det A \neq 0$
- think of $\det A$ = volume of parallelepiped spanned by column (or row) vectors



volume $= 0 \Longleftrightarrow$ row vector linearly dependent

$\Longleftrightarrow Ax = y$ does not have unique solution x for all y

$\Longleftrightarrow A^{-1}$ does not exist

- det def. by Leibniz or Laplace formula
- $\det(A \cdot B) = \det A \det B \quad \left(\Rightarrow 1 = \det A^{-1} A = \det A^{-1} \det A \Rightarrow \det A^{-1} = \frac{1}{\det A} \right)$