Session 3
Sep. 11, 2019

Ex.: $f(x) = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}$

$$\Rightarrow Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$\Rightarrow \det Df(x,y) = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x} > 0 \quad \forall (x,y) \Rightarrow Df$ everywhere non-singular

$\Rightarrow$ inverse fct. thm. applies

but note that f is not globally invertible (periodic in y!)

1.2 Review of Topology

<u>Def.</u>: Let X be a set, $\tau = \{U_i \subset X\}_{i \in I}$ (I some index set) with

- $\emptyset, X \in \tau$
- <u>arbitrary</u> unions of U_i's $\in \tau$
- <u>finite</u> intersections of U_i's $\in \tau$

Then each U_i is called open set, each $U_i^c = X \setminus U_i$ closed set, τ a topology, (X, τ) a topological space, any $U_i \ni p$ a (open) neighborhood of p

Ex.: metric topology on a metric space (X,d)

↳ def. open balls $B_r(x) = \{y \in X : d(x,y) < r\}$ as open

↳ $U \subset X$ is open if $\forall x \in U\ \exists r > 0$ with $B_r(x) \subset U$

[metric d: • $d(x,y) \geq 0$
• $d(x,y) = 0 \iff x = y$
• $d(x,y) = d(y,x)$
• $d(x,z) \leq d(x,y) + d(y,z)$]

$\Rightarrow$ allows us to define

- convergence: $(x_i)_{i \geq 0} \to x$ if for every neighborhood U of x $\exists N \in \mathbb{N}$ s.t. $x_i \in U\ \forall i \geq N$

- continuity: preimages of open sets are open

Def.: A bijection $f: X \to Y$ with f and f^{-1} continuous is called homeomorphism.

we often want to study topologies with more structure

Def.: (X,τ) is called Hausdorff if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there are (open) neighborhoods U_1 of x_1, U_2 of x_2 with $U_1 \cap U_2 = \emptyset$.

Ex.: • metric topology is Hausdorff (choose $x_1, x_2 \in X$, $\delta = d(x_1,x_2) \Rightarrow U_1 = B_{\delta/3}(x_1)$, $U_2 = B_{\frac{\delta}{3}}(x_2)$)

• Zariski (cofinite) topology on $\mathbb{R}$ (or $\mathbb{C}$): U open $\iff U = \emptyset$ or $X \setminus U$ is finite

↳ not Hausdorff

Generating topologies, basis:

Def.: Take any set X and $\mathcal{B}$ a collection of subsets of X with

(a) $X = \bigcup_{B \in \mathcal{B}} B$,

(b) $\forall B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2$ $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Then set of all unions of elements of $\mathcal{B}$ is called the topology generated by $\mathcal{B}$.

note: • $\emptyset$ is taken to be included in (b)

• it is indeed a topology by def. and due to (b) (finite intersections included)

• alternatively to (b) we could just include finite intersections

Ex.: open balls in $\mathbb{R}^n$ generate standard topology

Def.: A collection $\mathcal{B} = \{\text{open sets of } X\}$ is a basis for (X, τ) if every open subset of X is the union of elements from $\mathcal{B}$.

Def.: (X, τ) is called second-countable if there is a countable basis for τ.

Is $\mathbb{R}^n$ second-countable? Yes, take balls at rational points with rational radius

Def.: A collection of (open) subsets of X s.t. their union is X is called (open) cover.

(For $S \subset X$, an open cover of S is a collection of open sets $\{U_i\}_{i \in I}$ s.t. $S \subset \bigcup_{i \in I} U_i$, I some index set.)

A subcollection that is still a cover is called subcover.

Thm.: Let (X, τ) be second-countable. Then every open cover of X has a countable subcover (= Lindelöf space).

Proof: next time