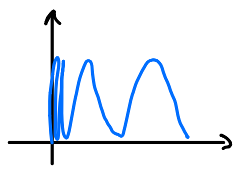
Session 5
Sep. 18, 2019

Recall: path-connected means: cont. path γ connects x and y $\forall x,y$

note: • path conn. $\Longrightarrow$ conn.

• for open subsets of $\mathbb{R}^n$: path conn. $\Longleftrightarrow$ conn.

• example for X that is conn. but not path conn.: topologist's sine curve $\{(x, \sin\frac{1}{x}) : x \in (0,1]\}$
$\cup \{(0,0)\}$



$\longrightarrow$ no path from 0 to rest of curve

• a maximal connected subset of X is called connected component of X

some results: • $I \subset \mathbb{R}$ connected $\Longleftrightarrow$ I interval or point

• $f: X \to Y$ cont., X connected $\Longrightarrow$ $f(X)$ connected

($f(X)$ not conn. $\Rightarrow \exists V \subset f(X)$ open and closed and $\neq \emptyset, \neq f(X) \Rightarrow$ same for $f^{-1}(V)$ by cont.
$\Rightarrow$ contradiction)

• $f: X \to \mathbb{R}$ cont., X conn., suppose $\exists\, a, b \in X$ s.t. $f(a) < 0 < f(b)$

$\Longrightarrow \exists\, c \in X$ s.t. $f(c) = 0$ (X conn. $\Rightarrow f(X)$ conn. $\Rightarrow f(X) =$ interval)

2. Manifolds: Definition and Examples

2.1 Topological Manifolds

manifold: looks locally like $\mathbb{R}^n$

Def.: A topological manifold M of dimension n is a Hausdorff and second-countable topological space s.t. every point in M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.

($\forall p \in M\ \exists$ open $U \subset M$, $p \in U$, and open $V \subset \mathbb{R}^n$, and homeomorphism $\varphi: U \to V$)

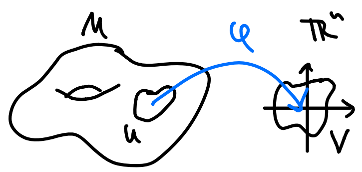
note:

- equivalently: could use homeomorphic to some open ball in $\mathbb{R}^n$ (or even unit ball in $\mathbb{R}^n$)

  Why? let $\varphi: U \to V$ as above $\Rightarrow \exists\, r > 0$ s.t. $B_r(\varphi(p)) \subset \Phi(U)$

  $\Rightarrow$ use $\varphi: \varphi^{-1}(B_r(\varphi(p))) \to B_r(\varphi(p))$

  (unit ball by rescaling)



- the dimension of a manifold is a topological invariant: an n-dim. manifold is never homeomorphic to an m-dim manifold for $m \neq n$

Def.: A pair (U, φ) with $U \subset M$ open, homeomorphism $\varphi: U \to V$ for open $V = \varphi(U) \subset \mathbb{R}^n$ is called (coordinate) chart. Also, we call:

- φ a (local) coordinate map,
- $\varphi(p) = (x^1(p), \dots, x^n(p))$ local coordinates,
- $\varphi^{-1}: V \to U$ a coordinate system.

Examples: • any open subset of $\mathbb{R}^n$ is a top. n-manifold

• n-sphere $S^n := \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1\}$

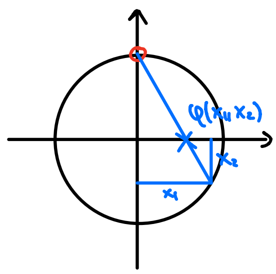
↳ Hausdorff and second countable clear

(note: Hausdorff and second countability generally transfer to subsets with subspace top.)

↳ charts: e.g. use stereographic projections

$$\varphi^+ : S^n \setminus \underbrace{\{(0,\dots,0,1)\}}_{\text{north pole}} \to \mathbb{R}^n, \quad \varphi^+(x^1,\dots,x^{n+1}) = \frac{1}{1-x^{n+1}}(x^1,\dots,x^n)$$

$$\varphi^- : S^n \setminus \underbrace{\{(0,\dots,0,-1)\}}_{\text{south pole}} \to \mathbb{R}^n, \quad \varphi^-(x^1,\dots,x^{n+1}) = \frac{1}{1+x^{n+1}}(x^1,\dots,x^n)$$



$$\varphi(x_1, x_2) = \frac{x_1}{1-x_2}$$

$\varphi^\pm$ are both homeomorphisms and their domains cover S^n

$\Rightarrow S^n$ is top. n-manifold

Thm.: Let $M_1, \dots, M_k$ be top. manifolds of dim. $n_1, \dots, n_k$. Then $M_1 \times \dots \times M_k$ is a top. manifold of dim. $n_1 + \dots + n_k$.

Proof: Hausdorff and second-countable follows directly for product topology.

Locally like $\mathbb{R}^n$: • for each $(p_1, \dots, p_k) \in M_1 \times \dots \times M_k$ choose corresponding charts (U_i, φ_i)

$\Rightarrow \varphi_1 \times \dots \times \varphi_k : U_1 \times \dots \times U_k \to \mathbb{R}^{n_1 + \dots + n_k}$ is homeomorphism onto its image □

Ex.: n-torus $\mathbb{T}^n = S^1 \times \dots \times S^1$