Session 6
Sep. 19, 2019

<u>Thm.:</u> Let M be a top. manifold. Then

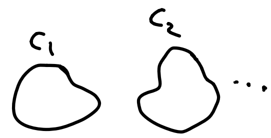
a) Every connected component is open, thus a top. manifold

b) M connected $\Leftrightarrow$ M path connected

<u>Proof:</u> a) M has basis of coordinate balls by def.

note: locally path-connected means: $\exists$ basis of path-conn. open subsets

$\hookrightarrow$ choose $x \in M$, say $x \in$ connected component C_1



$\hookrightarrow$ choose open neighborhood U of x that is homeomorphic to some open ball in $\mathbb{R}^n$

$\Rightarrow U$ connected $\Rightarrow U \subset C_1 \Rightarrow C_1$ open

b) path-conn. $\Rightarrow$ conn. ✓

conn. + locally path conn. $\Rightarrow$ path conn. $\square$

another example of a manifold: n-dim. real projective space $\mathbb{P}^n$

$\hookrightarrow$ recall: $\sim$ is an equivalence relation means:
- $x \sim x$ (reflexive)
- $x \sim y \Rightarrow y \sim x$ (symmetric)
- $x \sim y$ and $y \sim z \Rightarrow x \sim z$ (transitive)

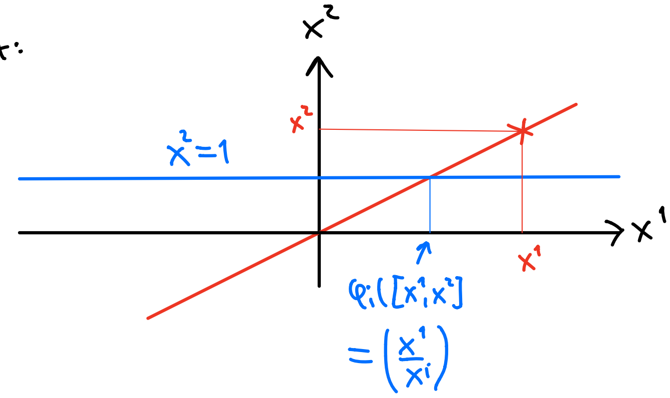
equivalence class $[x] := \{y : x \sim y\}$

$\hookrightarrow$ here we def. for $x, y \in \mathbb{R}^{n+1}$ that $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$

$\Rightarrow \mathbb{P}^n =$ set of all equivalence classes (= all straight lines through origin = all 1-dim. linear subspaces of $\mathbb{R}^n$)

def. natural map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, $\pi(x) = [x]$

construction of a chart:



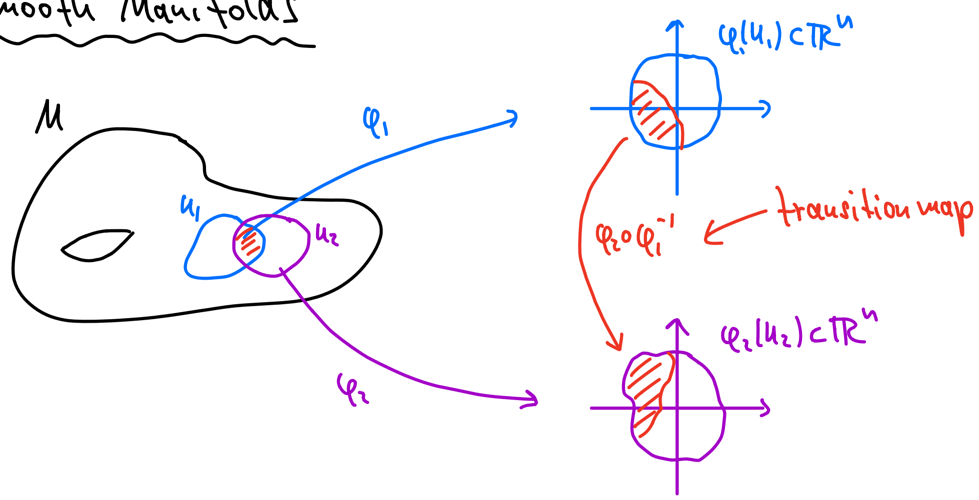
Let $\hat{U}_i = \{ y \in \mathbb{R}^{n+1} \setminus \{0\} : x^i \neq 0 \}$ and $U_i = \pi(\hat{U}_i)$

$\Rightarrow \varphi_i : U_i \to \mathbb{R}^n,\ \varphi_i([x^1, \dots, x^{n+1}]) = \frac{1}{x^i}(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1})$

with $\varphi_i^{-1}(u^1, \dots, u^n) = [u^1, \dots, u^{i-1}, 1, u^i, \dots, u^n]$, both φ_i and φ_i^{-1} are continuous

since $U_1, \dots, U_{n+1}$ cover $\mathbb{P}^n$ (+ Hausdorff + second countable), $\mathbb{P}^n$ is an n-manifold

2.2 Smooth Manifolds



for a sensible def. of smooth fct.'s on M, we need a smooth structure on M

Def.: Let M be a top. n-manifold. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ for some index set I, s.t.

- U_α are open and cover M,
- $\forall \alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\varphi_\beta \varphi_\alpha^{-1} : \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\subset \mathbb{R}^n,\text{ open}} \to \underbrace{\varphi_\beta(U_\alpha \cap U_\beta)}_{\subset \mathbb{R}^n,\text{ open}}$ is C^r ($(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^r compatible)

Then $\mathcal{A}$ is called a C^r atlas for M, and $(M, \mathcal{A})$ a C^r manifold.