Session 7
Oct. 1, 2019

last time we defined that $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is a C^r atlas of a manifold M if

- $\{U_\alpha\}$ open cover for M
- $\varphi_\beta \varphi_\alpha^{-1}$ is C^r on $U_\alpha \cap U_\beta$ $\forall \alpha, \beta$

$(M, \mathcal{A})$ is called C^r manifold

note:
- smooth atlas/manifold $= C^\infty$ atlas/manifold
- just atlas means C^0 atlas ($= \{U_\alpha\}$ open cover)
- in general atlas is not unique (starting with a top. manifold)
- any C^k atlas is also a C^ℓ atlas if $k > \ell$
- how to check that (U, φ) and (V, ψ) are C^r compatible (i.e., $\psi \circ \varphi^{-1}$ a C^r diffeomorphism)?
  ↳ check if $\psi \circ \varphi^{-1}$ is C^r and injective and Jacobian non-singular
  $\Rightarrow$ C^r compatible by inverse fct. thm.

non-uniqueness usually not a problem:

<u>Def.:</u> A C^r (differentiable) structure on M is a maximal C^r atlas $\mathcal{A}$ (i.e., not contained in any larger C^r atlas).

note:
- maximal C^r atlas = union of all C^r-equivalent atlases ($\mathcal{A}, \mathcal{A}'$ are C^r equivalent if $\mathcal{A} \cup \mathcal{A}'$ is C^r atlas)
- one can show: any C^r atlas is contained in exactly one maximal C^r atlas (but a top. manifold might have many such maximal C^r atlases)
- furthermore: for every C^r structure $\exists$ unique C^r-equivalent C^∞ structure
  $\Rightarrow$ usually we consider smooth manifolds only

Some hard problems: • top. manifold that does not admit any smooth structure (ex. by Kervaire 1960)

• how many smooth structures does n-sphere have?

↳ $n=1,2,3,5,6$: one
↳ $n=4$: unknown
↳ $n=7$: 28 (first ex.: Milnor 1956)

Examples:

• $\mathbb{R}^n$ is a smooth n-manifold, e.g., choose one chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$ (standard smooth structure on $\mathbb{R}^n$)

• product smooth manifold structure

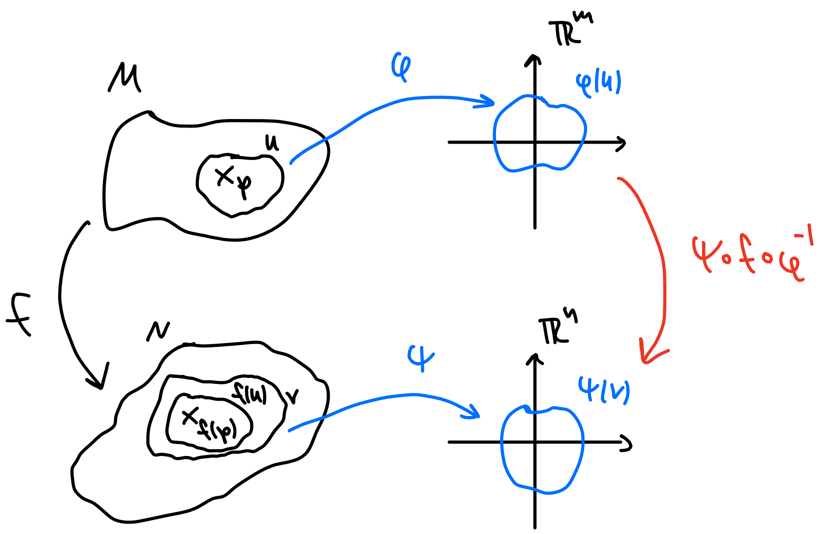
• open subset $U \subset M$ of smooth manifold $(M, \mathcal{A})$

$\Rightarrow \mathcal{A}_U := \{(V \cap U, \varphi|_{V\cap U}) : (V, \varphi) \in \mathcal{A}\}$, then $(U, \mathcal{A}_U)$ is a smooth manifold

• sphere S^n: use either stereographic projections or direct projections, see HW

• real projective space $\mathbb{P}^n$: see HW

2.3 Smooth Maps between Manifolds



<u>Def.:</u> Let M, N be smooth manifolds. A map $f: M \to N$ is smooth at $p \in M$ if there are charts (U, φ) with $p \in U$ and (V, ψ) with $f(p) \in V$ s.t. $f(U) \subset V$ and $\underset{\subset \mathbb{R}^m}{\psi \circ f \circ \varphi^{-1}: \varphi(U)} \to \underset{\subset \mathbb{R}^n}{\psi(V)}$ is smooth at $\varphi(p)$.

note:
- f smooth $\iff$ f smooth $\forall p \in M$
- $f: M \to \mathbb{R}^k$ smooth $\Rightarrow$ $N = \mathbb{R}^k$, and can choose $V = \mathbb{R}^k$, $\psi = id$ in def.
- smoothness property independent of choice of chart due to def. of smooth atlas
- simple ex.s: identity $id: M \to M$ and constant maps $f: M \to M$ are smooth