Session 8
Oct. 2, 2019

recall: $p \in U$, $f(p) \in V$, charts (U, φ), (V, ψ) s.t. $f(U) \subset V$

then $f: M \to N$ is smooth in $p \in M$ if $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$ in the usual sense

(i.e., as fct. from an open set in $\mathbb{R}^m$ to an open set in $\mathbb{R}^n$)

Standard results:

<u>Proposition:</u> f smooth $\Rightarrow$ f continuous

<u>Proof:</u> Notation as in def., then $f|_U = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi : U \to V$ is cont. as composition of cont. fct.s $\Rightarrow$ f cont. by HW 2 Problem 1. $\square$

Standard statements with straightforward proof:

<u>Proposition:</u>

- $f_i : M \to N_i$ smooth $\Rightarrow$ $f: M \to N_1 \times \cdots \times N_k$, $f(p) = (f_1(p), \dots, f_k(p))$ smooth
- $f: M \to N$ and $g: N \to P$ smooth $\Rightarrow$ $g \circ f : M \to P$, $(g \circ f)(p) = g(f(p))$ smooth
- $f, g : M \to \mathbb{R}^n$, $\lambda : M \to \mathbb{R}$ smooth $\Rightarrow$ $f + g$, λf, and $\langle f, g \rangle$ smooth

$\langle f, g \rangle : M \to \mathbb{R}$, $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle = \sum_{i=1}^{n} f_i(x) g_i(x)$

Classification of smooth manifolds:

<u>Def.:</u> Let M, N be smooth manifolds. A homeomorphism $f: M \to N$ s.t. f and f^{-1} are smooth is called a diffeomorphism. M and N are called diffeomorphic ($M \approx N$) if there exist a diffeomorphism $f: M \to N$.

note: • $M \approx M$ (use id.)

• $M \approx N \Rightarrow N \approx M$ (by def.)

• $M \approx N$ (with diffeomorphism f), $N \approx P$ (with diffeom. g) $\Rightarrow M \approx P$ (using $f \circ g$)

$\Rightarrow$ equivalence class of diffeomorphic manifolds

↳ classification of 3-manifolds is a current research direction

↳ ex.: any compact connected 1-manifold is diffeomorphic to S^1

Ex.: • $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $\mathbb{R}^n$ with standard smooth structure

↳ take $f: \mathbb{R}^n \to \mathbb{B}^n$, $f(x) = \frac{x}{\sqrt{1+\|x\|^2}}$ $\Rightarrow$ smooth

↳ inverse $f^{-1}: \mathbb{B}^n \to \mathbb{R}^n$, $f^{-1}(y) = \frac{y}{\sqrt{1-\|y\|^2}}$ $\Rightarrow$ smooth

$\Rightarrow \mathbb{B}^n$ diffeomorphic to $\mathbb{R}^n$

Homeomorphism invariance of dim. requires an advanced proof, but diffeomorphism invariance is easier:

Let $f: M \to N$ be a diffeomorphism $\Rightarrow \psi \circ f \circ \varphi^{-1}: \underset{\subset \mathbb{R}^m}{\varphi(U)} \to \underset{\subset \mathbb{R}^n}{\psi(V)}$ is a diffeomorphism $\Rightarrow m = n$

Diffeomorphism invariant properties are a central research subject

2.4 Partitions of Unity

tool to "glue together" local smooth objects into global ones

<u>Def.:</u> Let $r_1 < r_2 \in \mathbb{R}$.

• a smooth fct. $h: \mathbb{R} \to \mathbb{R}$ with $h(x) = \begin{cases} 1, & x \leq r_1 \\ \text{between 0 and 1}, & r_1 < x < r_2 \\ 0, & x \geq r_2 \end{cases}$ is called cutoff fct.

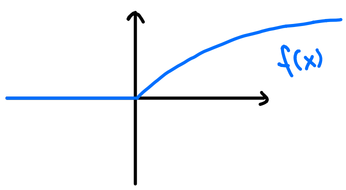
- a smooth fct. $H:\mathbb{R}^n \to \mathbb{R}$ with $H(x) = \begin{cases} 1 & \text{on } \overline{B_{r_1}(0)} \\ \text{between } 0 \text{ and } 1 & \text{on } B_{r_2}(0) \setminus \overline{B_{r_1}(0)} \\ 0 & \text{on } \mathbb{R}^n \setminus B_{r_2}(0) \end{cases}$

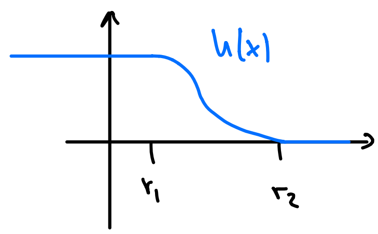
  is called (smooth) bump fct.

Such fct.s exist:

e.g., def. $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}} & , x>0 \\ 0 & , x \le 0 \end{cases}$, from Analysis we know that f is smooth

Then $h(x) = \dfrac{f(r_2 - x)}{f(r_2-x)+f(x-r_1)}$ is a smooth cutoff fct. , $H(x) = h(\|x\|)$ a smooth bump fct.





Note/recall: $\operatorname{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}$ (support of f)

next: glueing together for manifolds

<u>Def.:</u> Let $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M. We call $\{\psi_\alpha : M \to \mathbb{R} \text{ cont. (smooth)}\}_{\alpha \in A}$ a (smooth) partition of unity subordinate to $\mathcal{X}$ if:

- $0 \le \psi_\alpha(x) \le 1 \quad \forall \alpha \in A, x \in M$
- $\operatorname{supp} \psi_\alpha \subset X_\alpha \quad \forall \alpha \in A$
- $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$ locally finite (each $x \in M$ has neighborhood U that intersects finitely many $\operatorname{supp}\psi_\alpha$'s)
- $\sum_{\alpha\in A} \psi_\alpha(x) = 1 \quad \forall x \in M$

<u>Thm.:</u> For a smooth manifold M and any open cover $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ there exists a smooth partition of unity subordinate to $\mathcal{X}$.

We skip the proof and refer to HW for applications.