Session 10
Oct. 9, 2019

Def.: Let M be a smooth manifold, $p \in M$. A linear map $D: C^\infty(M) \to \mathbb{R}$ is called **derivation at p** if $D(fg) = f(p) Dg + g(p) Df \quad \forall f,g \in C^\infty(M)$ (smooth $M \to \mathbb{R}$).

$T_pM = \{D : D \text{ derivation}\}$ is called **tangent space to M at p**.

Note: T_pM is a vector space

Def.: Let M, N be smooth manifolds, $F: M \to N$ smooth, $p \in M$. The **differential of F at p** or **push forward** is a map $dF_p : T_pM \to T_{F(p)}N$ def. by

$$\underbrace{\underbrace{dF_p(\underbrace{v}_{\in T_pM})}_{\in T_{F(p)}N}(\underbrace{f}_{\in C^\infty(N)})}_{\in \mathbb{R}} = \underbrace{v(\underbrace{f \circ F}_{M \to \mathbb{R}})}_{\in \mathbb{R}} \qquad \forall v \in T_pM,\ f \in C^\infty(N)$$

Note: $dF_p(v)$ is indeed linear (v derivation) and a derivation at $F(p)$:

$$\begin{aligned} dF_p(v)(fg) &= v((fg)\circ F) = v((f\circ F)\cdot(g \circ F)) \\ &= (f \circ F)(p)\, v(g \circ F) + (g\circ F)(p)\, v(f \circ F) \\ &= f(F(p))\, dF_p(v)(g) + g(F(p))\, dF_p(v)(f) \end{aligned}$$

Properties: M, N, P smooth manifolds, $F: M \to N$, $G: N \to P$ smooth, $p \in M$, then
(proofs omitted)

- $dF_p : T_pM \to T_{F(p)}N$ is linear
- $d(G \circ F)_p : T_pM \to T_{G\circ F(p)}P$, $d(G\circ F)_p = \underbrace{dG_{F(p)}}_{T_{F(p)}N \to T_{G\circ F(p)}P} \circ \underbrace{dF_p}_{T_pM \to T_{F(p)}N}$

see HW

- $d(\underbrace{Id_M}_{\text{identity}})_p = Id_{T_pM}$
- If F diffeomorphism $\Rightarrow$ dF_p isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

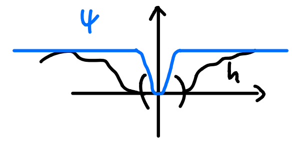
Important: tangent spaces are really local!

<u>Proposition:</u> M smooth manifold, $U \subset M$ open, $i: U \to M$ inclusion map. Then $\forall p \in U$,

$di_p : T_pU \to T_pM$ is a canonical isomorphism.

$\Rightarrow$ $di_p(v)$ and v are really "the same"

Proof uses this Proposition: let $p \in M$, $v \in T_pM$. If for $f,g \in C^\infty(M)$, $f|_U = g|_U$ for some neighborhood U of p then $vf = vg$.



we just prove this: • $h := f - g$, so $h|_U = 0$

• ψ = bump fct. with $\psi = 1$ on $\operatorname{supp}(h)$, $\operatorname{supp}\psi \subset M \setminus \{p\}$

$\Rightarrow \psi h = h \Rightarrow h(p) = \psi(p) = 0 \Rightarrow v(h\psi) = h(p)\, v\psi + \psi(p)\, vh = 0$

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$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad v(h) \Rightarrow vf = vg$ $\square$

<u>Corollary:</u> T_pM has same dimension as M.

↳ proven by using a local smooth chart at p.

How to do computations?

choose smooth chart (U, φ) at p $\Rightarrow d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ isomorphism

$\uparrow$

basis $\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$

$$\Rightarrow \left.\frac{\partial}{\partial x^i}\right|_p = \underbrace{(d\varphi_p)^{-1}}_{=(d\varphi^{-1})_{\varphi(p)}} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \text{ basis of } T_pM$$

note: $f \in C^\infty(U) \Rightarrow \left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1})$

$$= \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

(recall def. $dF_p(v)(f) = v(f \circ F)$)

$\hat{f} = f \circ \varphi^{-1}$, $\hat{p} = \varphi(p)$ coordinate representations of f and p

So what is dF_p in local coordinates (for $F: M \to N$)? $\longrightarrow$ see HW