Session 12
Oct. 16, 2019

3.2 Submanifolds

recall: M smooth m-manifold with atlas $\mathcal{A}$, $U \subset M$ open

$\Rightarrow$ atlas $\mathcal{A}_U = \{ (V \cap U, \varphi|_{V \cap U}) : (V, \varphi) \in \mathcal{A} \}$

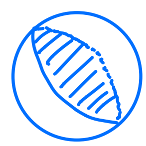
$\Rightarrow$ U is also a smooth m-manifold, called open submanifold of M

we want to consider more general submanifolds, e.g. donut as submanifold of $\mathbb{R}^3$

note: • identify $\mathbb{R}^k$ with $\mathbb{R}^n$, $k < n$: $\{ (x^1, \dots, x^k, x^{k+1}, \dots, x^n) : x^{k+1} = \dots = x^n = 0 \} \subset \mathbb{R}^n$

• a k-slice of open $U \subset \mathbb{R}^n$ is $\{ (x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U : x^{k+1} = c^{k+1}, \dots, x^n = c^n \}$

<u>Def.:</u> Let N^n be a smooth manifold, $M \subset N$. M is called embedded submanifold of dimension $m \leq n$ if $\forall p \in M$ there is a coordinate chart (V, ψ) of N, $p \in V$, $\psi(p) = 0$, s.t.



$$\psi(M \cap V) = \underbrace{\{ (x^1, \dots, x^m, x^{m+1}, \dots, x^n) \in \psi(V) : x^{m+1} = \dots = x^n = 0 \}}_{m\text{-slice of } \psi(V) \subset \mathbb{R}^n}$$

note: • M is indeed a manifold (with the subspace topology)

• one can show that inclusion map $i: M \hookrightarrow N$ is an embedding (hence the name)

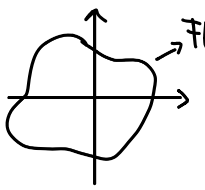
Next: how to characterize embedded submanifolds?

A) images of certain immersions

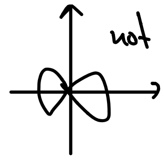
B) level sets $F^{-1}(\{q\}) \subset M$ for $F: M \to N$, $q \in N$

Recall: $F: M \to N$ smooth

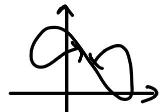
- (smooth) immersion: dF_p injective $\forall p$



$\to F(M)$ submanifold



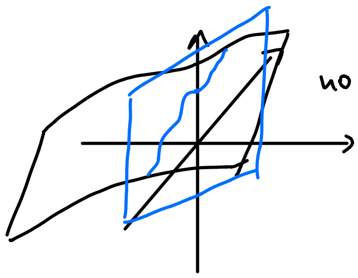
not subm.

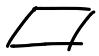


not subm

$\Rightarrow F(M)$ submanifold under some conditions

- (smooth) submersion: dF_p surjective $\forall p$



no $dF_p =$  (plane parallel to x-y plane

$\Rightarrow$ level sets $F^{-1}(\{q\})$ submanifolds

$\hookrightarrow$ should also be true if F is not a submersion, but dF_p surjective $\forall p \in F^{-1}(\{q\})$

A)

Proposition: If $F: M \to N$ is an embedding, then $F(M)$ is an embedded submanifold of N, and $F: M \to F(M)$ a diffeomorphism.

Proof: consider $q = F(p)$, centered charts (U, φ) at p, (V, ψ) at q s.t. $F(U) \subset V$

Rank-thm. for embedding $F \Rightarrow \psi \circ F \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$

now: $F(U) \subset F(M)$ open $\Rightarrow \exists$ neighborhood $W \ni q$ s.t. $F(U) = F(M) \cap W$

take $W \subset V \Rightarrow \psi \circ F(U) = \psi(F(M) \cap W) =$ m-slice of $\psi(W)$.

Diffeomorphism: basically clear from def.: F^{-1} smooth since F immersion $\square$

Proposition: If $F: M \to N$ is a smooth injective immersion and M compact, then $F(M)$ is an embedded submanifold.

Proof: M compact, N Hausdorff $\Rightarrow F: M \to N$ maps open sets into open sets $\square$

B)

Proposition: If $F: M \to N$ be smooth with constant rank r, $q \in N$, then $F^{-1}(\{q\})$ is an embedded submanifold of M of dimension $\dim M - r$.

Proof: let $p \in F^{-1}(q)$, choose centered charts (U, φ) and (V, ψ) containing p and q

$$\text{Rank Thm.} \Rightarrow \psi \circ F \circ \varphi(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, \underset{\overset{\uparrow}{n}}{0})$$

$$\Rightarrow F^{-1}(q) \cap U = \{(0, \dots, 0, x^{r+1}, \dots, x^m)\} \Rightarrow m-r \text{ slice} \qquad \square$$

note: in particular: F submersion $\Rightarrow F^{-1}(\{q\})$ embedded submanifold

Next: need only check surjectivity of dF_p for $p \in F^{-1}(\{q\})$