Session 13
Oct. 17, 2019

Def.: Let $F: M \to N$ be smooth.

- If dF_p is surjective for some $p \in M$, p is a regular point of F; otherwise p is a critical point of F.
- If all $F^{-1}(\{q\})$ are regular points, $q \in N$ is called regular value; if not, q is called critical value.

note: $\dim M < \dim N \Rightarrow$ all $p \in M$ are critical points

Proposition: If $F: M \to N$ smooth, $q \in F(M)$ a regular value, then $F^{-1}(\{q\})$ is an embedded submanifold of dimension $\dim M - \dim N$.

Proof: similar to before by Rank Thm.

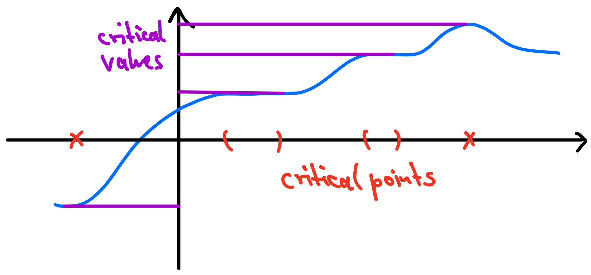
Ex.: n-sphere: consider $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $F(x^1, \dots, x^{n+1}) = \sum_{j=1}^{n+1} (x^j)^2$

$$\Rightarrow dF_x = 2(x^1, \dots, x^{n+1})$$

$$\Rightarrow \operatorname{rank} dF_x = 1 \text{ for } x \neq 0 \Rightarrow \text{any } x \neq 0 \text{ is a regular point}$$

$$\Rightarrow F^{-1}(\{q\}) \text{ for any } \mathbb{R} \ni q \neq 0 \text{ is an } n\text{-dim. embedded submanifold of } \mathbb{R}^{n+1}$$

3.3 Sard's Theorem



Sard: critical values have measure 0

In this section we only consider $\mathbb{R}^n$

Recall from Analysis II:

box in $\mathbb{R}^n$: $R = [a_1, b_1] \times \dots \times [a_n, b_n]$

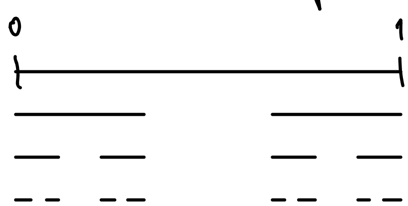
volume of R = Lebesgue measure $\lambda(R) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$

Def.: Any $A \subset \mathbb{R}^n$ has Lebesgue measure zero if for any $\varepsilon > 0$ there exist countable boxes $R_1, R_2, \dots$ such that $A \subset \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \lambda(R_i) < \varepsilon$.

note: could also take balls instead of boxes

Ex.: • A countable has measure 0 (choose points as boxes)

• Cantor set:



start with $[0,1]$, always cut out the middle thirds

$$\Rightarrow \text{volume} = 1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots\right)$$
$$= 1 - \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$
$$= 1 - \frac{1}{3} \frac{1}{1 - \frac{2}{3}}$$
$$= 0$$

but Cantor set is actually uncountable

HW sheet 5:

Problem 1: a) not so hard

b) recall: $T_a\mathbb{R}^n = \{\text{all derivations}\}$

↳ very abstract: all we have is linearity and product rule

we know any directional derivative $D_v|_a$ is a derivation

but does the other way around also hold?

this is the question here

If answered, we know that $\frac{\partial}{\partial x^1}\big|_a, \dots, \frac{\partial}{\partial x^n}\big|_a$ is a basis of $T_a\mathbb{R}^n$

(but do not use this for this exercise)

Problem 2: curves $\gamma : \underset{\text{take this}}{(-1,1)} \to M$, $\gamma(0) = p$

forget about tangent spaces, then we don't know what a derivative of γ is!

but: $f\circ\gamma : [0,1] \to \mathbb{R}$

therefore we say: if $(f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$ $\forall f \in C^\infty(M)$ then $\gamma_1 \sim \gamma_2$

$\mathcal{V}_p M = \{[\gamma]\}$ = space of all derivatives, still abstract

Note: $f: M \to N$, want to def. derivative $d_pf : \mathcal{V}_pM \to \mathcal{V}_{f(p)}N$

How? Take $[\gamma] \in \mathcal{V}_pM$, consider new curve $\tilde{\gamma} = f\circ\gamma : (-1,1) \to N$

$\Rightarrow \tilde{\gamma}(0) = f(p)$

$\Rightarrow$ def. $d_pf[\gamma] := [f\circ\gamma]$

In class we defined tangent spaces differently:

$T_pM = \{\text{all } w : C^\infty(M) \to \mathbb{R},\ w \text{ linear, + product rule at } p\}$

Claim of this exercise is that those two def.s can be translated into each other by the map

$$\Psi : V_pM \longrightarrow T_pM \ , \ [\gamma] \longmapsto \underbrace{d\gamma_0}_{\text{map } T_0(-1,1) \to T_pM} (\partial_x)$$

$$d\gamma_0(\partial_x)f := \partial_x (f \circ \gamma)(x)\big|_0 = (f \circ \gamma)'(0)$$