3.3 Sard's Theorem

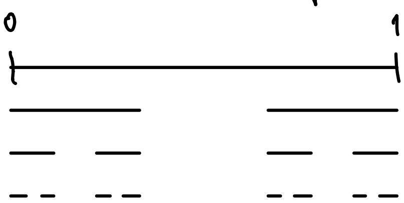
Session 14
Oct. 29, 2019

recall:

<u>Def.:</u> Any $A \subset \mathbb{R}^n$ has lebesgue measure zero if for any $\varepsilon > 0$ there exist countable boxes $R_1, R_2, \ldots$ such that $A \subset \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \lambda(B_i) < \varepsilon$.

Ex.: • A countable has measure 0 (choose points as boxes)

• Cantor set:



start with $[0,1]$, always cut out the middle thirds

$$\Rightarrow \text{volume} = 1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \ldots\right)$$
$$= 1 - \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$
$$= 1 - \frac{1}{3} \frac{1}{1 - \frac{2}{3}}$$
$$= 0$$

but Cantor set is actually uncountable

<u>Lemma:</u> Countable unions of sets of measure zero have measure zero.

Proof: Let $\varepsilon > 0$, call the sets of measure zero A_i.

Choose boxes $R_{i,1}, R_{i,2}, \ldots$ to cover A_i s.t. $\sum_{j=1}^{\infty} \lambda(R_{i,j}) < \frac{\varepsilon}{2^i}$

$\Rightarrow \{R_{i,j}\}_{i,j \in \mathbb{N}}$ covers $\bigcup_{i \in \mathbb{N}} A_i$

$$\Rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda(R_{i,j}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon. \qquad \square$$

Sard's Theorem: Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ be smooth. Then the set of critical values of f has Lebesgue measure zero.

Note: For $n > m$, this means $f(U)$ has measure zero.

Proof for $m<n$ and $m=n$:

$m<n$: idea: boxes in $\mathbb{R}^m$ are smaller than boxes in $\mathbb{R}^n$ e.g., $\bigcup_{n=3}^{\infty} [\frac{1}{n}, 1-\frac{1}{n}] = (0,1)$

from topology in $\mathbb{R}^m$: we know that we can write U as countable union of cubes

$$R = [a_1, a_1+\delta] \times \dots \times [a_m, a_m+\delta] \subset U$$

$\Rightarrow$ if $\lambda(f(R)) = 0$, we are done (by previous lemma)

mean-value thm.: $\|f(x)-f(y)\| \leq K \|x-y\| \quad \forall x,y \in R$ for some $K>0$

uniform in x,y, since R compact

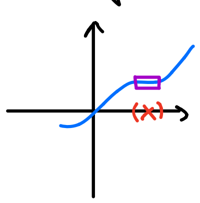
now choose $N \geq 1$, divide R into smaller cubes R_j^m with side length $\frac{\delta}{N}$ $\Rightarrow N^m$ cubes

$\Rightarrow f(R_j^m) \subset$ ball of radius $K\frac{\delta}{N}$, with volume $C\left(\frac{\delta}{N}\right)^n$ for some $C>0$.

$\Rightarrow f(R) \subset \bigcup_j R_j^m$, with volume $\leq N^m C\left(\frac{\delta}{N}\right)^n = C\delta^n N^{m-n}$

$\Rightarrow$ by choosing N large enough, $\lambda(f(R))$ can be made arbitrarily small

$m=n$: idea: image of a ball around critical point contained in small cylinder



choose cube R as above, $C := \{\text{critical points}\}$ ($f(C)$ = critical values), show $\lambda(f(R\cap C)) = 0$

↳ divide R into N^n rectangles (side length $\frac{\delta}{N}$) R_j^n

by def. of derivative: $\|f(x) - f(y) - Df(x)(x-y)\| \leq \varepsilon_N \|x-y\| \quad \forall x,y \in R_j^n$, and $\varepsilon_N \xrightarrow{N\to\infty} 0$

(see also HW 1, Problem 3)

$\leq \varepsilon_N C_n \frac{\delta}{N}$ for some C_n (e.g., $C_n = \sqrt{n}$)

next: fix $x \in R_j^n \cap C$ (in case this is non-empty)

then $Df(x)$ not surjective $\Rightarrow \{Df(x)(x-y) : y \in R_j^n\} \subset V$ = some $n-1$ dim. subspace of $\mathbb{R}^n$

$\Rightarrow \{f(x) - f(y) : y \in R_j^n\}$ has distance $\varepsilon_N C_n \frac{\delta}{N}$ from V

$\Rightarrow \{f(y) : y \in R_j^n\}$ has distance $\varepsilon_N C_n \frac{\delta}{N}$ from hyperplane $V + g(x)$

mean-value thm.: $\|f(x) - f(y)\| \leq K\|x-y\| \leq K C_n \frac{\delta}{N}$

$\Rightarrow \{f(y) : y \in R_j^n\} \subset$ cylinder of height $2\varepsilon_N C_n \frac{\delta}{N}$ and base = sphere of radius r,

with $r \leq K C_n \frac{\delta}{N}$

$\Rightarrow$ volume of cylinder $\leq C\left(\frac{\delta}{N}\right)^{n-1} \frac{\delta}{N} \varepsilon_N$ for some $C > 0$

$\Rightarrow f(R\cap C) \subset$ cylinder of volume $\leq C\left(\frac{\delta}{N}\right)^n \varepsilon_N \underbrace{N^n}_{\#\text{ of cubes } R_j^n} = C\delta^n \varepsilon_N$ $\square$

Remarks:
- C^1 is enough instead of smoothness in our proof of $m=n$, but for $m>n$, need C^k with $k > \max(0, m-n)$
  ($\exists$ example of $f : \mathbb{R}^2 \to \mathbb{R}$ C^1 with $f(C) \supset$ interval, due to Whitney)
- continuity not enough: space-filling curves $F : [0,1] \to [0,1]^2$ surjective
- Sard's thm. also true for manifolds (measure zero well-defined, bc. this property is diffeomorphism invariant, i.e., $A \subset M$ measure zero in one chart $\Rightarrow$ measure zero in all charts)