4. Lie Groups

Session 16
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connect groups and smooth manifolds

motivation: continuous symmetries of differential eq.s, e.g., Galilean symm. in classical mechanics (translation, rotation, uniform motion), or Poincaré symm. in relativity

recall:

sometimes denoted $x \cdot y$

Def.: A group is a set G with an operation $\cdot : G \times G \to G$, $(x,y) \mapsto xy$ that satisfies:

- $(xy)z = x(yz)$ (associativity)
- $\exists$ identity e, i.e. $ex = xe = x \;\; \forall x \in G$ (note: unique)
- $\exists$ inverse, i.e., $\forall x \in G \; \exists x^{-1} \in G$ with $x^{-1}x = xx^{-1} = e$ (note: unique)

If also $xy = yx \;\; \forall x,y \in G$, then G is called abelian group.

If $H \subset G$ with operation $\cdot$ is also a group, $(H, \cdot)$ is called a subgroup of G.

Def.: A Lie group is a smooth manifold G that is also a group, with the property that

- multiplication map $\cdot : G \times G \to G$, $(x,y) \mapsto xy$, and
- inverse map $G \to G$, $x \mapsto x^{-1}$

are smooth.

Examples: • additive group $(\mathbb{R}^n, +)$

↳ map $(x,y) \mapsto x+y$ is smooth

↳ map $x \mapsto -x$ is smooth

$\Rightarrow$ abelian connected Lie group of dimension n

- multiplicative group $(\mathbb{R}^*, \cdot)$, where $\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}$

  $\hookrightarrow$ maps $(x,y) \mapsto x \cdot y$ and $x \mapsto \frac{1}{x}$ are smooth $(x \neq 0)$

  note: – $\mathbb{R}^*$ is not connected: $\mathbb{R}^* = \underbrace{\mathbb{R}^{>0}}_{\mathbb{R}^+} \cup \underbrace{\mathbb{R}^{<0}}_{\mathbb{R}^-}$ union of disjoint open sets

  – $\mathbb{R}^{>0}$ is a subgroup of $\mathbb{R}^*$ and open, thus itself a Lie group

- general linear group $GL_n(\mathbb{R})$ = set of invertible $n \times n$ matrices with matrix multiplication

  note: – $M_{n\times n}(\mathbb{R})$ (real $n \times n$ matrices) is a vector space, thus a smooth manifold

  – $A \in GL_n(\mathbb{R}) \iff \det A \neq 0$

  $\Rightarrow$ since $\det: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is continuous, $GL_n(\mathbb{R})$ is open, thus also a smooth manifold

  – $(A,B) \mapsto A \cdot B$ smooth (matrix entries are polynomials)

  – $A \mapsto A^{-1} = \frac{1}{\det A} \underbrace{\operatorname{adj} A}_{\text{adjugate of } A\text{, some polynomial}}$ smooth

  $\Rightarrow GL_n(\mathbb{R})$ is a Lie group

- $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A > 0\}$

  $\hookrightarrow \det AB = \det A \cdot \det B$ and $\det A^{-1} = \frac{1}{\det A}$ $\Rightarrow GL_n^+(\mathbb{R})$ subgroup and open $(\det^{-1}((0,\infty)))$

  $\Rightarrow$ Lie group

- similar: $GL(V)$ for vector space V is a Lie group

- circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with complex multiplication is a Lie group

  $\Rightarrow$ n-torus $S^1 \times \ldots \times S^1$ is an n-dim. Lie group

- Heisenberg group $H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$ plays important role in physics

  one can check that it is a Lie group

Def.: Let H, G be Lie groups. A smooth map $F: G \to H$ is called Lie group homomorphism if

$$F(\underbrace{xy}_{\text{mult. in } G}) = \underbrace{F(x)F(y)}_{\text{mult. in } H} \quad \forall x,y \in G.$$

If F also a diffeomorphism, we call it Lie group isomorphism.

Ex.: • $\exp: \mathbb{R} \to \mathbb{R}^*$, $x \mapsto e^x$ Lie group homomorphism

- $\exp: \mathbb{R} \to \mathbb{R}^+$ Lie group isomorphism
- $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$ Lie group homomorphism
- Lie group G, $g \in G$, then conjugation by g is the map $C_g: G \to G$, $h \mapsto ghg^{-1}$
  ↳ smooth, group homomorphism and isomorphism $(C_g^{-1} = C_{g^{-1}})$
  ↳ Def.: a subgroup $H \subset G$ is called normal if $C_g(H) = H \ \forall g \in G$.