Session 17
Nov. 7, 2019

Recall: • $F: G \to H$ Lie group homomorphism: $F(xy) = F(x)F(y) \ \forall x,y \in G$

• subgroup $H < G$ is called normal if $C_g(H) = H \ \forall g \in G$ $\quad (C_g(h) = ghg^{-1})$

<u>Thm.:</u> Every Lie group homomorphism has constant rank.

Proof: use isomorphism property and chain rule, HW

basic properties of Lie groups:

<u>Proposition:</u> Let G^0 be the identity component of G (the connected component containing the identity). Then G^0 is a Lie group, $\dim G^0 = \dim G$, and G^0 is a normal subgroup of G.

Proof: HW

<u>Def.:</u> Let $g \in G$, G Lie group, then the left and right translations $L_g, R_g : G \to G$ are defined as $L_g(h) = gh$, $R_g(h) = hg$.

<u>Proposition:</u> L_g and R_g are diffeomorphisms.

Proof: smooth as composition of smooth maps, $L_{g^{-1}}, R_{g^{-1}}$ smooth inverses. $\square$

In some later class: • $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ as Lie groups

• representation theory: representation = Lie group homomorphism $G \to GL(V)$

↳ e.g., $\Psi(x)$ solution to a translation or rotation invariant eq.,

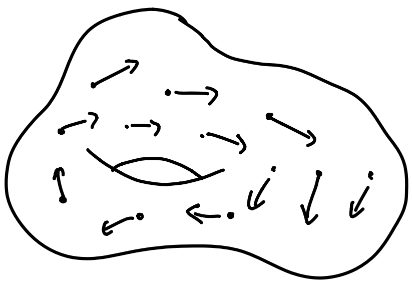
$\Psi \in$ some vector space (also function spaces, Hilbert spaces)

$\Rightarrow \Psi(Rx)$, $R \in G$ is a different solution

$\Psi(Rx) = \underbrace{\Pi(R)}_{\text{representation}} \Psi(x)$ $\quad \Rightarrow$ important research topic

5. Vector Fields and Differential Forms

5.1 Vector Fields



attach vector to each $p \in M$
$\downarrow$
$\in T_pM$

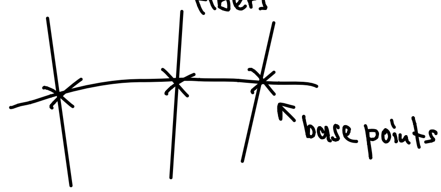
Def.: Let M be a smooth manifold, then the tangent bundle is def. as

$$TM := \bigsqcup_{p \in M} T_pM = \{(p,v) : p \in M, v \in T_pM\}$$

disjoint union $\left(= \dot{\bigcup_{p\in M}} T_pM \right)$

Note: • projection $\pi : TM \to M$, $\pi(p,v) = p$ = base point

$\pi^{-1}(p)$ called fiber



• simple case: $T\mathbb{R}^n = \bigsqcup_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \mathrel{\hat{=}} \bigsqcup_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

but usually no canonical identification of different T_pM's

Proposition: Let M be a smooth n-manifold, then TM has a natural topology and smooth structure that make it into a $2n$-dim. smooth manifold. Furthermore, the projection π is smooth.

Proof idea: • take chart (U, φ), $p \in U$

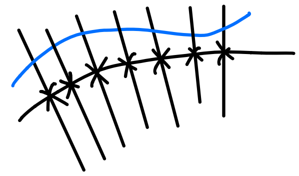
• def. $\tilde{\varphi} : \underset{TM}{\pi^{-1}(U)} \to \mathbb{R}^{2n}$, $\tilde{\varphi}(p,v) = (\varphi(p), \overbrace{d\varphi_p(v)}^{T_pM \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n}) \in \varphi(U) \times \mathbb{R}^n$

• smoothness of transition maps can be checked $\square$

Def.: A smooth vector field is a smooth map $X: M \to TM$, s.t. $X(p) \in T_pM \ \forall p \in M$

(i.e., $\pi \circ X = id_M$).

note: • X is a called section of the map $\pi: TM \to M$



Ex.: • zero section $X(p) = 0_{T_pM}$, not a constant map (0_{T_pM} can vary for different p)

• open $U \subset \mathbb{R}^n$, $T_pU \cong \mathbb{R}^n$, def. vector fields $\frac{\partial}{\partial x^i}(p) = e_i$ (e_i = i-th canonical basis vector of $\mathbb{R}^n$)

$\Rightarrow \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ basis of $T_pU \ \forall p \in U$

any vector field can be written as $X(p) = \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}(p)$, f^i smooth