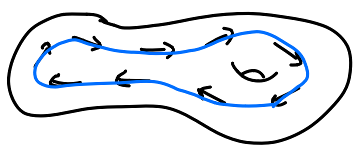
Session 19
Nov. 14, 2019

5.2 Integral Curves



M smooth manifold, $I \subset \mathbb{R}$ open interval

curve $\gamma : I \to M$

$\Rightarrow$ velocity at $t_0 = \dot{\gamma}(t_0) = \gamma'(t_0) := d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}M$

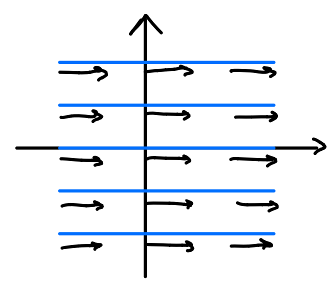
i.e., $\gamma'(t_0) f = d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) f = \frac{d}{dt}\Big|_{t_0} (f\circ\gamma) = \underbrace{(f\circ\gamma)'}_{\substack{\text{derivative of}\\ f\circ\gamma : I \to \mathbb{R}}}(t_0)$

in local coordinates: $\gamma'(t_0) = \sum_i \frac{d\gamma^i}{dt}(t_0) \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)}$

Def.: A smooth curve $\gamma : I \to M$ is called integral curve of the vector field $X \in \mathfrak{X}(M)$

if $\gamma'(t) = X(\gamma(t)) \quad \forall t \in I$.

Ex.: • $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x^1} \Rightarrow X(\gamma(t)) = \frac{\partial}{\partial x^1}\Big|_{\gamma(t)}$



$$\gamma'(t) = \frac{d\gamma^1}{dt}\frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \frac{d\gamma^2}{dt}\frac{\partial}{\partial x^2}\Big|_{\gamma(t)}$$

$\Rightarrow$ integral curves are $\gamma(t) = \begin{pmatrix} a+t \\ b \end{pmatrix}$ for $a, b \in \mathbb{R}$

• $M = \mathbb{R}^2$, $X = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$

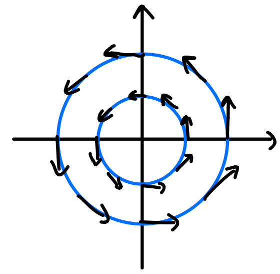
$$\Rightarrow \dot{\gamma}^1(t) \frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \dot{\gamma}^2(t)\frac{\partial}{\partial x^2}\Big|_{\gamma(t)} = \gamma^1(t)\frac{\partial}{\partial x^2}\Big|_{\gamma(t)} - \gamma^2(t)\frac{\partial}{\partial x^1}$$

$\Rightarrow$ need to solve system of two ODEs: $\dot{\gamma}^1(t) = -\gamma^2(t)$

$\dot{\gamma}^2(t) = \gamma^1(t)$

solution: $\gamma(t) = \begin{pmatrix} a\cos t - b\sin t \\ a\sin t + b\cos t \end{pmatrix}$, circles

(and $\gamma(t) = (0,0)$)



$\Rightarrow$ Finding integral curves = solving system of ODEs in local coordinates:

$$\dot{\gamma}^i(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = X^i(\gamma(t)) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

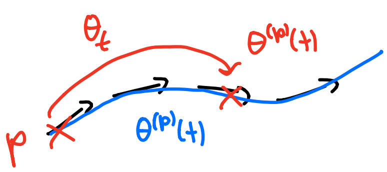
$$\Rightarrow \quad \dot{\gamma}^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)) \quad , i = 1, \dots, n \qquad \text{(autonomous ODEs)}$$

<u>Proposition:</u> Let $X \in \mathfrak{X}(M)$ for a smooth manifold M. Then $\forall p \in M$ there is $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of X with $\gamma(0) = p$.

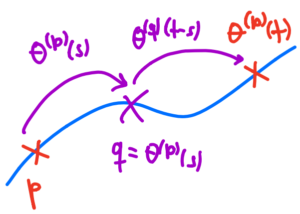
Proof notes:
- this is a classical result for $M = \mathbb{R}^n$, which is proved using the Banach fixed point theorem
- since this is a local result only, this proves the proposition for any M by choosing local coordinates

next: consider curve $\theta^{(p)}(t)$ starting at p $(\theta^{(p)}(0) = p)$

now fix t, def. $\theta_t : M \to M$, $\theta_t(p) = \theta^{(p)}(t)$



note:



$\Rightarrow \quad \theta_s \theta_{t-s} = \theta_t \quad , \quad$ or $\quad \theta_t \circ \theta_s = \theta_{t+s}$

Def.: $\theta : \mathbb{R} \times M \to M$ smooth is called global flow if $(\theta(t,p) =: \theta_t(p))$

↳ or "one-parameter group action"

- $\theta_0(p) = p \quad \forall p \in M$
- $\theta_t(\theta_s(p)) = \theta_{t+s}(p) \quad \forall p \in M,\ t,s \in \mathbb{R}$

The map $X : M \to TM$, $X(p) = \underbrace{\theta^{(p)\prime}(0)}_{\text{velocity at } t=0 \text{ of curve with starting point } p}$ is called infinitesimal generator of θ.

One can show that this X is indeed a smooth vector field, and $\theta^{(p)}$ are its integral curves.

Ex.: • $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x} \Rightarrow$ flow $\tau : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\tau_t(x,y) = \begin{pmatrix} x+t \\ y \end{pmatrix}$

But if $M = \mathbb{R}^2 \setminus \{0\}$ flow is not global.

Def.: $\theta : (-\varepsilon, \varepsilon) \times U \to M$ smooth is called local flow if
$U \underset{\text{open}}{\subset} M$

- $\theta_0(p) = p \quad \forall p \in U$
- $\theta_t(\theta_s(p)) = \theta_{t+s}(p)$.

whenever this exists

Fundamental Theorem on Flows: For any $X \in \mathfrak{X}(M)$ $\exists$ local flow θ, s.t. $\theta(p)$ are the integral curves of X starting at $p \in M$.

the flow generated by X

Def.: $X \in \mathfrak{X}(M)$ is called complete if it generates a global flow.

Proposition: For compact smooth manifolds M, any vector field is complete.

Proof sketch: compactness $\Rightarrow$ finite cover $\Rightarrow$ patch together local domains of flows.