Session 20
Nov. 20, 2019

recall:
- vector field $X: M \to TM$ s.t. $X(p) \in T_pM$
- global derivation $D: C^\infty(M) \to C^\infty(M) \iff Df = Xf$ for some $X \in \mathfrak{X}(M)$
- Lie bracket $[X,Y]$ is a derivation
- integral curve $\gamma'(t) = X(\gamma(t)) \to$ local existence
- flow $\theta: (-\varepsilon,\varepsilon) \times \underset{\overset{\cap}{M}}{U} \to M$, $\theta(t,p) = \theta_t(p)$ s.t.
  - $\theta_0(p) = p$
  - $\theta_t \circ \theta_s = \theta_{t+s}$
- local flows for vector fields always exist, global flows not always

↳ i.e., $\theta_\cdot(p)$ are the integral curves (starting at p) for vector field

Next: recall that in local coordinates $(Xf)(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x^i}\Big|_p$

↳ what about directional derivatives of vector fields?

want sth. like $\lim_{t\to 0} \frac{X(p+tv) - X(p)}{t}$ on $\mathbb{R}^n$

↳ on M: replace $X(p+tv)$ by $X(\theta_t(p))$, for some flow θ of vector field Y

but still $X(p) \in T_pM$, $X(\theta_t(p)) \in T_{\theta_t(p)}M$, so how to identify tangent spaces?

$\underset{\text{fixed } t}{\theta_t}: M \to M$, so $d(\theta_t)_p : T_pM \to T_{\theta_t(p)}M$, and $\left(d(\theta_t)_q\right)^{-1} = d(\theta_t^{-1})_q = d(\theta_{-t})_q$

maps $T_{\underbrace{\theta_t(p)}_{q}}M \to T_pM$

$\Rightarrow$ We def. the Lie derivative of X with respect to Y as

$$\mathcal{L}_Y X(p) = \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(X(\theta_t(p))) - X(p)}{t} = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}(X(\theta_p(t))),$$

where θ is the flow of Y

<u>Thm.:</u> $\mathcal{L}_Y X = [Y,X]$ (proof skipped: with some more background it can be reduced to a direct computation)

5.3 Covectors

V a (finite dim. real) vector space

Def.: Any linear map $\omega: V \to \mathbb{R}$ (i.e., real-valued linear functional) is called covector.

$V^* =$ dual space of $V = \{$all covectors$\}$

Ex.: $V = \{$column vectors in $\mathbb{R}^n\}$, then $V^* = \{$row vectors in $\mathbb{R}^n\}$

e.g. $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\omega = (\omega^1, \dots, \omega^n) \Rightarrow \omega(v) = (\omega^1, \dots, \omega^n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \omega^i v_i$

$= \omega^i v_i$

Einstein summation convention:
summation implied if same index appears twice (as an upper and lower index

$e_1, \dots, e_n$ basis of $V \Rightarrow \varepsilon^1, \dots, \varepsilon^n$ basis of V^*, i.e., $\varepsilon^i(e_j) = \delta^i_j$ $\quad (\varepsilon^i(v) = v^i)$

in general:

Def.: $\varepsilon^1, \dots, \varepsilon^n \in V^*$ called dual basis to basis $E_1, \dots, E_n$ of V if $\varepsilon^i(E_j) = \delta^i_j$

Prop.: Dual basis is indeed a basis of V^*. (Proof: easy, linear algebra)

$\Rightarrow$ for $V^* \ni \omega = \omega_i \varepsilon^i$, $V \ni v = v^j E_j$ we have $\omega(v) = \omega_i v^j \varepsilon^i(E_j) = \omega_i v^i$

Def.: Let V, W be vector spaces, $A: V \to W$ linear, then dual map $A^*: W^* \to V^*$ is def. by
$(A^*\omega)(v) = \omega(Av)$ for all $\omega \in W^*$, $v \in V$.

Note:
- $(A \circ B)^* = B^* \circ A^*$
- $\exists$ canonical isomorphism $V \to V^{**}$ (canonical = independent of arbitrary choices, e.g., of basis)
- $\exists$ isomorphism $V \to V^*$, but it is not canonical (since it is basis dependent)