Session 22
Nov. 27, 2019

last time: A and $B: \underbrace{V\times\dots\times V}_{k \text{ times}} \to \mathbb{R}$ multilinear $\Rightarrow$ def. tensor product $A\otimes B$

- $\mathcal{B} = \{\varepsilon^{i_1}\otimes\dots\otimes\varepsilon^{i_k} : 1\le i_1\le n, \dots, 1\le i_k\le n\}$ basis of $L(V,\dots,V;\mathbb{R}) \cong V^*\otimes\dots\otimes V^*$

  $\{\varepsilon^i\}_{i=1,\dots,n}$ some (dual) basis of V^*

<u>Def.:</u> $T^k(V^*) = \underbrace{V^*\otimes\dots\otimes V^*}_{k \text{ times}}$ is called space of covariant k-tensors (or covariant rank-k tensors)

$\Lambda^k(V^*)$ = alternating (antisymmetric) covariant k-tensors, i.e., for $\alpha\in\Lambda^k(V^*)$ we have

$$\alpha(\dots,v_i,\dots,v_j,\dots) = -\alpha(\dots,v_j,\dots,v_i,\dots)$$

<u>Def.:</u> Alternation $\mathrm{Alt}: T^k(V^*) \longrightarrow \Lambda^k(V^*)$, def. by

$$(\mathrm{Alt}\,\alpha)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

sign of the permutation σ: $+1$ for σ even, -1 for σ odd

S_k: symmetric group (all permutations of $1,\dots,k$)

Ex.: $\beta\in T^2(V^*) \Rightarrow (\mathrm{Alt}\,\beta)(v_1,v_2) = \frac{1}{2}\big(\beta(v_1,v_2) - \beta(v_2,v_1)\big)$

<u>Def.:</u> For $\omega\in\Lambda^k(V^*)$, $\eta\in\Lambda^\ell(V^*)$, their wedge product (or exterior product) is def. as

$$\Lambda^{k+\ell}(V^*) \ni \omega\wedge\eta = \underbrace{\frac{(k+\ell)!}{k!\,\ell!}}\,\mathrm{Alt}(\omega\otimes\eta)$$

combinatorial factor included for convenience

Properties of wedge product (straightforward to check):

- bilinear, associative
- $\omega\wedge\eta = (-1)^{k\ell}\,\eta\wedge\omega$

- $\{\varepsilon^{i_1}\wedge\ldots\wedge\varepsilon^{i_k} : i_1<\ldots<i_k\}$ is a basis of $\Lambda^k(V^*) \Rightarrow \dim \Lambda^k(V^*) = \binom{n}{k}$ $(n=\dim V)$
- $\omega^1\wedge\ldots\wedge\omega^k(v_1,\ldots,v_k) = \det(\omega^j(v_i))$

Next: from k-tensors to k-1 tensors

Def.: For $v\in V$, we define the interior multiplication

$$i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*),\ \omega \mapsto i_v\omega = v \lrcorner\, \omega \quad (v \text{ into } \omega) \text{ s.t.}$$

$$\underbrace{(i_v\omega)}_{\in\Lambda^{k-1}(V^*)}(v_1,\ldots,v_{k-1}) = \omega(v,v_1,v_2,\ldots,v_{k-1})$$

Properties:
- $i_v\circ i_v = 0$ (since ω's above are alternating)
- $\omega\in\Lambda^k(V^*)$, $\eta\in\Lambda^\ell(V^*) \Rightarrow i_v(\omega\wedge\eta) = (i_v\omega)\wedge\eta + (-1)^k\,\omega\wedge(i_v\eta)$ (direct computation)

5.5 Differential Forms

Def.: $\Lambda^k(T^*M) = \bigsqcup_{p\in M} \Lambda^k(T_p^*M)$ = alternating covariant k-tensor bundle on M

A differential k-form is an alternating covariant (smooth) tensor field $M\to\Lambda^k(T^*M)$. We denote $\Omega^k(M) := \{\text{all } k\text{-forms}\}$.

Remarks:
- $\omega\in\Omega^k(M)$, $\eta\in\Omega^k(M)$, then $\omega\wedge\eta$ def. pointwise, i.e., $(\omega\wedge\eta)_p = \omega_p\wedge\eta_p$
- in coordinates: $\omega = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}\, dx^{i_1}\wedge\ldots\wedge dx^{i_k} \equiv \sum_I{}' \omega_I\, dx^I$, $I=\{i_1,\ldots,i_k\}$ a multi-index

  smooth functions: $\omega_{i_1\ldots i_k} = \omega\left(\frac{\partial}{\partial x^{i_1}},\ldots,\frac{\partial}{\partial x^{i_k}}\right)$

Ex.s on $\mathbb{R}^3$: $\omega = \sin(xy)\,dy\wedge dz$, $\eta = dx\wedge dy\wedge dz$

Def.: $F: M \to N$ smooth, $w \in \Omega^k(N)$, then the pullback $F^*\omega \in \Omega^k(M)$ is def. as

$$\underbrace{(F^*\omega)_p}_{\in \Lambda^k(T_p^*M)}(v_1, \dots, \underbrace{v_k}_{\in T_pM}) = \underbrace{\omega_{F(p)}}_{\in \Lambda^k(T^*_{F(p)}N)}(dF_p(v_1), \dots, d\underbrace{F_p(v_k)}_{T_{F(p)}N})$$