Session 23
Nov. 28, 2019

last time: $\cdot\ \Omega^k(M) \ni$ differential forms $\omega: M \to \Lambda^k(T^*M)$ smooth

s.t. $\omega_p \in \Lambda^k(T_p^*M)$

$\cdot$ in coordinates: $\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}\, dx^{i_1}\wedge\dots\wedge dx^{i_k} \equiv \sum_I{}' \omega_I\, dx^I$

<u>Def.:</u> $F: M\to N$ smooth, $\omega\in\Omega^k(N)$, then the pullback $F^*\omega\in\Omega^k(M)$ is def. as

$$\underbrace{(F^*\omega)_p}_{\in\Lambda^k(T_p^*M)}(v_1,\dots,\underbrace{v_k}_{\in T_pM}) = \underbrace{\omega_{F(p)}}_{\in\Lambda^k(T^*_{F(p)}N)}(dF_p(v_1),\dots,\underbrace{dF_p(v_k)}_{T_{F(p)}N})$$

Rules for computation:

$\cdot\ F^*(\omega\wedge\eta) = (F^*\omega)\wedge(F^*\eta)$

$\cdot$ in coordinates: $F^*\left(\sum_I{}' \omega_I\, dy^{i_1}\wedge\dots\wedge dy^{i_k}\right) = \sum_I{}' (\omega_I\circ F)\, d(y^{i_1}\circ F)\wedge\dots\wedge d(y^{i_k}\circ F)$

$\cdot$ M, N smooth n-manifolds, (x^i) some local coordinates on M, (y^i) on N, $u: V\to\mathbb{R}$, $V \subset N$, then

$$F^*(u\, dy^1\wedge\dots\wedge dy^n) = (u\circ F)\det \underbrace{Jac(F)}_{\text{Jacobian matrix of } F}\, dx^1\wedge\dots\wedge dx^n$$

Examples: see HW

recall: $f\in C^\infty(M) = \Omega^0(M)$ (a 0-form), then the differential df is a 1-form $(\in\Omega^1(M))$

next generalize this to a map $d: \Omega^k(M)\to\Omega^{k+1}(M)$

Consider $M=\mathbb{R}^n$ first, want to generalize, e.g., curl $\frac{\partial\omega_j}{\partial x^i} - \frac{\partial\omega_i}{\partial x^j}$

$\Rightarrow$ this defines a 2-form $d\omega = \sum_{i<j}\left(\frac{\partial\omega_j}{\partial x^i} - \frac{\partial\omega_i}{\partial x^j}\right) dx^i\wedge dx^j$

for any $\omega = \sum_J{}' \omega_J\, dx^J$ (k-form on $\mathbb{R}^n$) we def. the exterior derivative

$$d\omega = d\Big(\sum_J{}' \omega_J dx^J\Big) = \sum_J{}' \underbrace{d\omega_J}_{\text{differential of } \omega_J:\, \mathbb{R}^n \to \mathbb{R}} \wedge dx^J = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

This def. has 4 properties, and we take these to generalize the def. to any smooth manifold M.

<u>Thm.:</u> The exterior differentiation $d: \Omega^k(M) \to \Omega^{k+1}(M)$ $\forall k$ is uniquely def. by the following properties:

a) d is $\mathbb{R}$-linear

b) $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

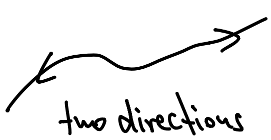
c) $d \circ d = 0$

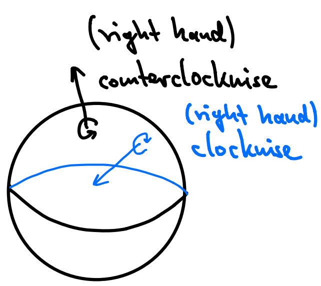
d) For $f \in C^\infty(M) = \Omega^0(M)$, df is the differential (def. by $df(X) = Xf$)

Proof can be found in Lee's book. HW: prove these properties for $M = \mathbb{R}^n$

Important property: for $F: M \to N$ smooth, $\omega \in \Omega^k(N)$: $F^*(d\omega) = d(F^*\omega)$

5.6 Orientation





first: n-dim. vector space V

<u>Def.:</u> Two bases $(E_1, \dots, E_n)$ and $(\tilde{E}_1, \dots, \tilde{E}_n)$ of V s.t. $E_i = B_i^{\,j} \tilde{E}_j$ are consistently ordered if $\det B > 0$.

An orientation $\mathcal{O}$ of V is an equivalence class $[E_1, \dots, E_n]$ of ordered bases.

V with a choice of orientation is called oriented vector space.

Proposition: $0 \neq \omega \in \Lambda^n(V^*)$ determines an orientation by setting

$$\mathcal{O}_\omega = [E_1, \dots, E_n] \text{ for } \omega(E_1, \dots, E_n) > 0.$$

Proof: use antisymmetry

on manifolds M:

pointwise orientation def. by choosing orientation of T_pM.

Def.: • local frame: (continuous) vector fields $E_1, \dots, E_n$ on $U \subset M$ s.t. $(E_1|_p, \dots, E_n|_p)$ basis of T_pM

• global frame: local frame on $U = M$

An orientation on M is a continuous pointwise orientation, i.e., $\forall p \in M$ $\exists$ oriented local frame (E_i) with $p \in U$ (= domain of (E_i))

There are orientable and nonorientable manifolds
↳ e.g., sphere ↳ e.g., Möbius strip

Proposition: $\omega \in \Omega^n(M)$ positively oriented $\Longleftrightarrow$ orientation on M

Proof: global version of the previous pointwise proposition.

orientation in terms of charts:

- A chart is positively oriented if the coordinate frame is.
- A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is consistently oriented if Jacobian of $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive determinant everywhere and $\forall \alpha, \beta$

Proposition: consistently oriented smooth atlas $\Longleftrightarrow$ orientation on M