Session 24
Dec. 3, 2019

Orientation: two equivalent approaches

- orientation of vector space: equivalence classes $[E_1, \dots, E_n]$ of consistently ordered bases (i.e., $\det B > 0$ if $E_i = B_i^{\,j} \tilde{E}_j$)

  orientation of manifold = continuous pointwise (i.e., of T_pM) orientation

  Proposition: $\omega \in \Omega^n(M)$ positively oriented $(\omega(E_1, \dots, E_n) > 0) \Longleftrightarrow$ orientation on M

- orientation in terms of charts:

  $\to$ chart is positively oriented if the coordinate frame $\left(\frac{\partial}{\partial x^i}\right)$ is

  $\to$ smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is consistently oriented if Jacobian of $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive determinant everywhere and $\forall \alpha, \beta$

Proposition: consistently oriented smooth atlas $\Longleftrightarrow$ orientation on M

Note: There are orientable and nonorientable manifolds

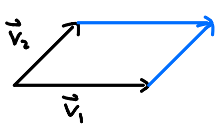
$\hookrightarrow$ e.g., sphere $\hookrightarrow$ e.g., Möbius strip

5.7 Integration on Manifolds

want coordinate-invariant def. of integration

heuristically: need "signed volume" at each $p \in M$: $v_1, \dots, v_k \in T_pM$ mapped to signed vol. $\omega(v_1, \dots, v_k)$

$\hookrightarrow$ recall:



volume $= \det(\vec{v}_1\, \vec{v}_2)$, also for higher dimension

$\hookrightarrow$ multilinear

$\hookrightarrow = 0$ if vectors linearly dependent

$\Rightarrow$ consider alternating forms

Start with 1-form ω on $[a,b] \subset \mathbb{R}$, i.e., $\omega_t = f(t)dt$

$$\Rightarrow \text{we def. } \boxed{\int_{[a,b]} \omega := \underbrace{\int_a^b f(t)dt}_{\text{usual Riemann (or Lebesgue) integral}}}$$

next: consider a domain of integration $D \subset \mathbb{R}^n$ (D bounded, ∂D measure 0)

let $\omega \in \Omega^n(\overline{D})$ (n-form), i.e., $\omega = f\, dx^1 \wedge \ldots \wedge dx^n$

↳ smooth (or cont. is enough)

$$\Rightarrow \text{we def. } \int_D \omega = \boxed{\int_D f\, dx^1 \wedge \ldots \wedge dx^n := \underbrace{\int_D f\, dx^1 \ldots dx^n}_{\text{Riemann int.}}} = \int_D f\, dV$$

<u>Proposition:</u> Let $D, E \subset \mathbb{R}^n$ be open domains of integration, $G: \overline{D} \to \overline{E}$ smooth (i.e., G can be continued to a smooth map $G: U \to V$, with U, V open) and orientation-preserving ($U \supset \overline{D}$, $V \supset \overline{E}$)

(dG_p: oriented bases of $T_pD \longrightarrow$ oriented bases of $T_{G(p)}E$)

diffeomorphism from $D \to E$, ω an n-form on $\overline{E}$. Then $\boxed{\int_D G^*\omega = \int_E \omega.}$

Proof: $(y^1, \ldots, y^n)$ coordinates on E, $(x^1, \ldots, x^n)$ coordinates on D, $\omega = f\, dy^1 \wedge \ldots \wedge dy^n$

$$\Rightarrow \int_E \omega := \int_E f\, dV \underset{\substack{\uparrow \\ \text{change of variables} \\ \text{for Riemann int.}}}{=} \int_D (f \circ G)\, |\det \underbrace{DG}_{\text{Jacobian}}|\, dV \underset{\substack{\uparrow \\ G \text{ orientation} \\ \text{preserving}}}{=} \int_D (f \circ G)(\det DG)\, dV$$

$$=: \int_D (f \circ G)(\det DG)\, dx^1 \wedge \ldots \wedge dx^n \underset{\substack{\uparrow \\ \text{pullback} \\ \text{formula} \\ \text{from last time}}}{=} \int_D G^*\omega \qquad \square$$

Note: $\int_D G^*\omega = -\int_E \omega$ if G orientation-reversing next: M a smooth oriented n-manifold

suppose n-form ω has compact support contained in one smooth chart (U,φ) (positively oriented).

<u>Def.:</u> The integral of ω over M is $\boxed{\int_M \omega := \int_{\varphi(U)} \underbrace{(\varphi^{-1})^* \omega}_{n\text{-form on } \mathbb{R}^n}}$

Independence of choice of chart: take (U,φ), $(\hat U,\hat\varphi)$, s.t. $\operatorname{supp}\omega \subset U \cap \hat U$

$\Rightarrow$ $\hat\varphi \circ \varphi^{-1}$ orientation-preserving diffeomorphism (if both charts are pos./neg. oriented)

$$\Rightarrow \int_{\hat\varphi(\hat U)} (\hat\varphi^{-1})^* \omega = \int_{\hat\varphi(\hat U\cap U)} (\hat\varphi^{-1})^* \omega \overset{\uparrow}{=} \int_{\varphi(\hat U \cap U)} (\hat\varphi\circ\varphi^{-1})^* (\hat\varphi^{-1})^* \omega = \int_{\varphi(\hat U\cap U)} (\varphi^{-1})^* (\hat\varphi)^* (\hat\varphi^{-1})^* \omega$$

previous proposition

$$= \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

next: • ω compactly supported n-form on M

• $\{U_i\}$ finite open cover of $\operatorname{supp}\omega$ (with positively oriented charts (U_i,φ_i))

• $\{\psi_i\}$ a smooth partition of unity (subordinate to $\{U_i\}$)

<u>Def.:</u> $\boxed{\int_M \omega := \sum_i \int_M \psi_i \omega}$

$\underbrace{\int_M \psi_i\omega}_{\text{well-def. on single chart } (U_i,\varphi_i)}$

<u>Proposition:</u> This def. is independent of the choice of open cover or partition of unity.

Properties: • linearity, orientation reversal, positivity

• diffeomorphism-invariance : $F: M \to N$ orientation-preserving diffeomorphism

$$\Rightarrow \int_M \omega = \int_N F^* \omega$$

Extra notes:

Example: $M = \mathbb{R}^2 \setminus \{0\}$, $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$, curve $\gamma : [0,2\pi] \to M$, $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\Rightarrow \gamma^* \omega = \frac{\cos t \underbrace{(d \sin t)}_{= \frac{\partial \sin t}{\partial t} dt} - \sin t \underbrace{(d \cos t)}_{= \frac{\partial \cos t}{\partial t} dt}}{(\cos t)^2 + (\sin t)^2} = (\cos t)^2 dt + (\sin t)^2 dt = dt$$

$$\Rightarrow \int_\gamma \omega = \int_{[0,2\pi]} \gamma^* \omega := \int_{[0,2\pi]} dt = 2\pi$$