Session 25
Dec. 4, 2019

we assume:

- M a smooth oriented n-manifold
- ω an n-form on M with $\operatorname{supp}\omega$ compact

If $\operatorname{supp}\omega$ covered by one chart (U,φ) then we def. $\boxed{\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega}$

This def. is independent of the choice of chart!

More generally, if

- $\{U_i\}$ finite open cover of $\operatorname{supp}\omega$ (with positively oriented charts (U_i,φ_i))
- $\{\psi_i\}$ a smooth partition of unity (subordinate to $\{U_i\}$)

<u>Def.:</u> $\boxed{\int_M \omega := \sum_i \int_M \psi_i \omega}$

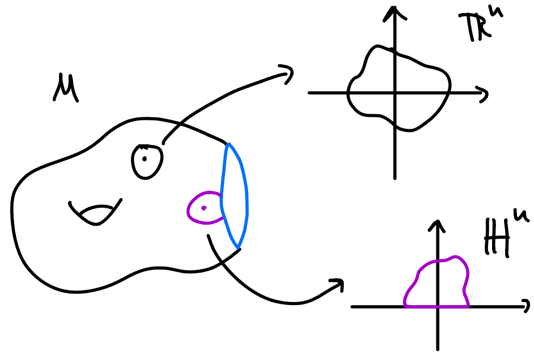
$\underbrace{\int_M \psi_i\omega}_{\text{well-def. on single chart } (U_i,\varphi_i)}$

<u>Proposition:</u> This def. is independent of the choice of open cover or partition of unity.

Properties:
- linearity, orientation reversal, positivity
- diffeomorphism-invariance: $F: N \to M$ orientation-preserving diffeomorphism

$$\Rightarrow \int_M \omega = \int_N F^*\omega$$

5.8 Manifolds with Boundary



Def.: $\mathbb{H}^n = \{(x^1,\dots,x^n) \in \mathbb{R}^n : x^n \geq 0\}$ (upper half space)

$\partial\mathbb{H}^n = \{(x^1,\dots,x^{n-1},0) \in \mathbb{R}^n\}$

An n-manifold with boundary is a second-countable Hausdorff space M where every $p \in M$ has a neighborhood homeomorphic to either an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$.

- $p \in M$ is a boundary point if it is in the domain of a boundary chart that maps p to $\partial\mathbb{H}^n$
- ∂M = set of all boundary points

Note:
- ∂M refers to manifold boundary, i.e., all $p \in M$ covered by a boundary chart
  $\to$ this is not necessarily equal to topological boundary (if we think of $M \subset$ some other topological space)
- $\partial M = (n-1)$-manifold without boundary (e.g., $\partial \overline{B^n} = S^{n-1}$)
- many results we discussed for manifolds without boundary also hold for manifolds with boundary

Proposition: If manifold with boundary M is orientable, then also ∂M is orientable (with a so-called induced orientation or Stokes orientation).

5.9 Stokes Theorem

Thm.: Let M be an oriented smooth n-manifold with boundary and ω a compactly supported smooth $(n-1)$-form on M. Then $\boxed{\int_M d\omega = \int_{\partial M} \omega}$ (Stokes Theorem)

Remarks:
- $d\omega$ = exterior derivative of ω = n form
- ∂M has orientation induced by M
- ω on right-hand side means $i_{\partial M}^* \omega$ ($i_{\partial M}: \partial M \to M$ inclusion)

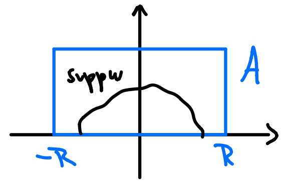
Corollary: M a compact oriented smooth manifold
- without boundary, then $\int_M d\omega = 0$ (η exact means $\eta = d\omega \Rightarrow \int_M$ exact form $= 0$)
- with boundary and ω closed (i.e., $d\omega = 0$), then $\int_{\partial M} \omega = 0$

Ex. in $\mathbb{R}^3$: $\int_V \underbrace{\operatorname{div} \vec{F}}_{\vec{\nabla}\cdot\vec{F}} \, dV = \int_{\partial V} \vec{F} \underbrace{d\vec{\sigma}}_{\vec{n}\, d\sigma}$ (clear if we establish connection of forms and vector fields clearly)

$\vec{n}$: outward pointing unit normal vector

Proof of Stokes:

simple case: $M = \mathbb{H}^n \Rightarrow \operatorname{supp}\omega \subset A = [-R,R]\times\dots\times[-R,R]\times[0,R]$



for some large enough R

general n-1 form: $\omega = \sum_i \omega_i \, dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n$

↳ hat means omitted

$$\Rightarrow d\omega = \sum_{i=1}^n \underbrace{d\omega_i}_{\sum_{j=1}^n \frac{\partial\omega_i}{\partial x^j}dx^j} \wedge dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i}\, dx^1\wedge\dots\wedge dx^n$$

$$\Rightarrow \int_{\mathbb{H}^n} d\omega = \sum_{i=1}^n (-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R \frac{\partial\omega_i}{\partial x^i}(x)\, dx^1\dots dx^n$$

$$= \sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\dots\underbrace{\int_{-R}^R \frac{\partial\omega^i}{\partial x^i}(x)\,dx^i}_{=\omega^i(x)\big|_{x^i=-R}^{x^i=R}=0\ (\operatorname{supp}\omega\subset A)} dx^1\dots\widehat{dx^i}\dots dx^n$$

$$+ (-1)^{n-1}\int_{-R}^R\dots\int_{-R}^R \underbrace{\int_0^R \frac{\partial\omega^n}{\partial x^n}(x)\,dx^n}_{\omega^n(x)\big|_{x^n=0}^{x^n=R} = -\omega^n(x^1,\dots,x^{n-1},0)} dx^1\dots dx^{n-1}$$

$$= (-1)^n \int_{-R}^R\dots\int_{-R}^R \omega^n(x^1,\dots,x^{n-1},0)\,dx^1\dots dx^{n-1}$$

and $$\int_{\partial\mathbb{H}^n}\omega = \sum_{i=1}^n\int_{A\cap\partial\mathbb{H}^n}\omega_i(x^1,\dots,x^{n-1},0)\,dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n$$

$$= \int_{A\cap\partial\mathbb{H}^n}\omega_n(x^1,\dots,x^{n-1},0)\,dx^1\wedge\dots\wedge dx^{n-1}$$

since $(x^1,\dots,x^{n-1})$ positively oriented for $\partial\mathbb{H}^n$ with even n (neg. for odd n), equality follows.

(induced orientation follows from first coordinate, whereas for manifolds with boundary last coordinate is zero $\Rightarrow (-1)^n$ factor)

general cases follow relatively directly from definitions of integral. □