Session 4
Sep. 12, 2019

1.3 Bonds

bond issuer (= borrower) pays interest and final payment to bondholder
= lender = buyer

↳ usually for long-term debts, e.g., issued by governments

↳ repaid at maturity date

Cashflow for level coupon bond: price/present value $P = \sum_{i=1}^{n \cdot m} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^{n \cdot m}}$

where:
- C = coupon payment
- r = interest rate
- F = par value
- n = # of periods (usually years)
- m = # interest compoundings per period
- with $C = \frac{F \cdot c}{m}$, c = coupon rate

$$\Rightarrow P = F \left(\sum_{i=1}^{nm} \frac{c/m}{(1+\frac{r}{m})^i} + \frac{1}{(1+\frac{r}{m})^{nm}} \right)$$

- yield to maturity = IRR = r given C, F, P, n, m

Ex.: 20 year, $\underbrace{9\%}_{=\text{coupon rate } c}$ bond, $\underbrace{\text{semiannual compounding}}_{\to m=2}$, interest rate $r=8\%$

$$\Rightarrow \text{price } P = F\left(\sum_{i=1}^{40} \frac{0.045}{(1.04)^i} + \frac{1}{(1.04)^{40}}\right) = 1.099 \cdot F$$

$\Rightarrow$ bond is sold at 109.9% of par

(e.g., par value $F = 1000\$ \Rightarrow P = 1099\$$ and $C = 45\$$)

with geom. series, $m=1$:

$$P = F\left(c \underbrace{\sum_{i=1}^{n} \frac{1}{(1+r)^i}}_{= -1 + \frac{1-(1+r)^{-(n+1)}}{1-(1+r)^{-1}} = \frac{-1+(1+r)^{-1}+1-(1+r)^{-(n+1)}}{1-(1+r)^{-1}}} + \frac{1}{(1+r)^n}\right)$$

$$= F\left(c\left(\frac{1-(1+r)^{-n}}{r}\right) + (1+r)^{-n}\right)$$

$$= F\left(\frac{c}{r} + \frac{(1-\frac{c}{r})}{(1+r)^n}\right)$$

terminology:

- $c = r$, then "bond sells at par"
- $c > r$, then "bond sells above par" or "at a premium"
- $c < r$, then "bond sells below par" or "at a discount"

note: often we use zero-coupon bonds, i.e., $C = 0$

$\Rightarrow$ single payment F $\quad \Rightarrow P = \dfrac{F}{(1+\frac{r}{m})^{nm}}$

1.4 Immunization

reduce risk from changes in the interest rate r if future liability L has to be met at period m (m called "horizon")

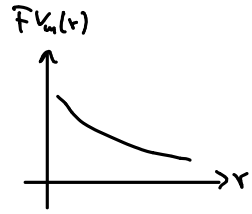
one could do simple cash-flow matching: buy zero-coupon bond with maturity m and par value $F = L$

but this has practical disadvantages
- → bond with exact maturity m might not exist
- → expensive
- → low yields

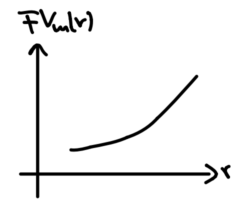
alternatively: consider a zero-coupon bond ($C=0$) with maturity n, par value F:

$$
\begin{aligned}
FV_m &= (1+r)^m P \\
&= (1+r)^m \frac{F}{(1+r)^n} \\
&= F\,(1+r)^{m-n}
\end{aligned}
$$

$n>m$:

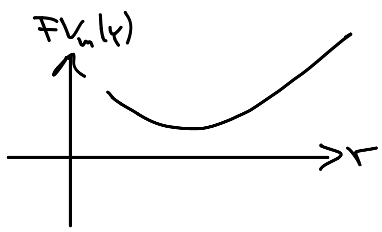


$n<m$:



now: set up portfolio with 2 zero-coupon bonds with maturities $n_1<m$, F_1 and $n_2>m$, F_2:

$$FV_m = F_1\,(1+r)^{m-n_1} + F_2\,(1+r)^{m-n_2} \overset{!}{=} L \quad \text{to meet liability}$$



to achieve stability w.r.t. changes in r $\Rightarrow$ find minimum of $FV_m(r)$