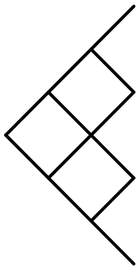
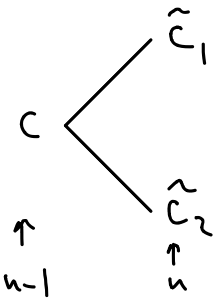
Session 9
Oct. 7, 2019

recall binomial tree model



one step



$$\underset{\substack{\uparrow\\ \text{option value}\\ \text{at step n}}}{c} = e^{-r\Delta t}\left(\overset{\substack{\text{risk-neutral probabilities}\\ \swarrow}}{p}\,\underset{\uparrow}{\tilde{c}_1} + (1-p)\,\underset{\nearrow}{\tilde{c}_2}\right)$$

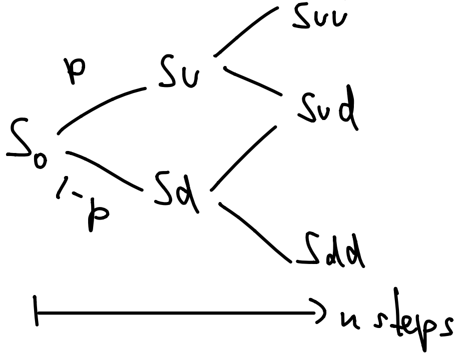
option value at step n

$$p = \frac{e^r - d}{u-d}$$

next, we world like to calibrate our model, i.e., choose u, d such that expectation value and variance converge as $n \to \infty$.

2.4 Binomial Tree and Calibration

recall: • model for stock price development



p is the probability for stock prices here, it is not the risk-neutral probability

recall: $\sum_{j=0}^{n} P(j,n) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j}$
$= (p+(1-p))^n = 1$

probability for j up's $=: P(j,n) = \binom{n}{j} p^j (1-p)^{n-j}$

now consider $S_T^{j\,up} = S_0 e^{Y_j}$ with stock's rate of return

$$Y_j = \ln \frac{S_T^{j\,up}}{S_0} = \ln u^j d^{n-j} = \ln\left(\left(\frac{u}{d}\right)^j d^n\right) = j \ln\left(\frac{u}{d}\right) + n \ln d$$

$\ln(ab) = \ln(a) + \ln(b)$, $\ln x^a = a \ln x$

next we want to compute expectation and variance of Y ($Y = Y_j$, fct. of j)

Def.: • Expectation value of X is $\mathbb{E}(X) = \sum_{j=0}^{n} X_j P(j,n)$

• Variance of X is $\operatorname{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$

Calculation rules:

- $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$ $(\lambda \in \mathbb{R})$

- $$\mathrm{Var}(x) = \mathbb{E}\left((x - \mathbb{E}(x))^2\right) = \mathbb{E}\left(x^2 - 2x\,\mathbb{E}(x) + \mathbb{E}(x)^2\right)$$

$$= \mathbb{E}(x^2) + \underbrace{\mathbb{E}(-2x\,\mathbb{E}(x))}_{= -2\,\mathbb{E}(x)^2} + \mathbb{E}(x)^2$$

$$= \mathbb{E}(x^2) - \mathbb{E}(x)^2$$

- $$\mathrm{Var}(\lambda x) = \lambda^2\,\mathrm{Var}(x)$$

- $$\mathrm{Var}(x+y) = \mathbb{E}((x+y)^2) - \mathbb{E}(x+y)^2$$

$$= \mathbb{E}(x^2 + 2xy + y^2) - \mathbb{E}(x)^2 - 2\,\mathbb{E}(x)\,\mathbb{E}(y) - \mathbb{E}(y)^2$$

$$= \underbrace{\mathbb{E}(x^2) - \mathbb{E}(x)^2}_{\mathrm{Var}(x)} + \underbrace{\mathbb{E}(y^2) - \mathbb{E}(y)^2}_{\mathrm{Var}(y)} + 2\underbrace{\left(\mathbb{E}(xy) - \mathbb{E}(x)\cdot\mathbb{E}(y)\right)}_{= \mathrm{Cov}(X,Y)}$$ (covariance of X and Y)

$\mathrm{Cov}(X,Y) = 0$ if x and y are independent

next: compute $\mathbb{E}(j)$, $\mathbb{E}(j^2)$:

- $$\mathbb{E}(j) = \sum_{j=0}^{n} j\,P(j,n) = \sum_{j=0}^{n} j \binom{n}{j} p^j (1-p)^{n-j} = \sum_{j=0}^{n} j \binom{n}{j} \left(\frac{p}{1-p}\right)^j (1-p)^n$$

$$\sum_{j=0}^{n} j \binom{n}{j} x^j = \sum_{j=1}^{n} j \frac{n!}{(n-j)!\,j!} x^j = \sum_{j=1}^{n} \frac{n!}{(n-j)!\,(j-1)!} x^j$$

$$= \sum_{j=1}^{n} n \underbrace{\frac{(n-1)!}{(n-1-(j-1))!\,(j-1)!}}_{=\binom{n-1}{j-1}} x^{j}$$

$$= n \sum_{j=1}^{n} \binom{n-1}{j-1} x^{j}$$

$$= nx \underbrace{\sum_{j=1}^{n} \binom{n-1}{j-1} x^{j-1}}_{= \sum_{j=0}^{n-1} \binom{n-1}{j} x^{j} = (1+x)^{n-1}}$$

$$= nx(1+x)^{n-1}$$

alternatively: $$\sum_{j=0}^{n} j \binom{n}{j} x^{j} = x \overbrace{\frac{d}{dx}\left(\sum_{j=0}^{n} \binom{n}{j} x^{j}\right)}^{\sum_{j=0}^{n} \binom{n}{j} j x^{j-1}}$$

$$= x \frac{d}{dx} (1+x)^{n}$$

$$= x n (1+x)^{n-1}$$

$$\Rightarrow \mathbb{E}(j) = n \underbrace{\left(\frac{p}{1-p}\right)}_{x} \Big(1 + \underbrace{\frac{p}{1-p}}_{x}\Big)^{n-1} (1-p)^{n}$$

$$= n \frac{p}{1-p} \left(\frac{1}{1-p}\right)^{n-1} (1-p)^{n}$$

$$= n \cdot p$$

- by similar computation: $\mathbb{E}(j^2) = np\big((n-1)p + 1\big)$

$$\Rightarrow \mathrm{Var}(j) = \mathbb{E}(j^2) - \mathbb{E}(j)^2 = np(1-p)$$

then we find: (recall $Y_j = j\ln(\frac{u}{d}) + n\ln d$)

- $\mathbb{E}(Y_j) = \mathbb{E}(j)\cdot\ln\frac{u}{d} + n\ln d = n\cdot p\cdot\ln\frac{u}{d} + n\ln d$

- $\mathrm{Var}(Y_j) = \mathrm{Var}(j\ln\frac{u}{d} + n\ln d) = (\ln\frac{u}{d})^2\,\mathrm{Var}(j) + \underbrace{\mathrm{Var}(n\ln d)}_{=0}$

  $= np(1-p)\,(\ln\frac{u}{d})^2$

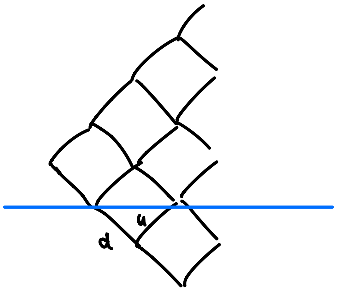
now: calibrate our model, meaning that we want

$$\mathbb{E}(Y_j) \xrightarrow{n\to\infty} \mu T \quad , \quad \mathrm{Var}(Y_j) \xrightarrow{n\to\infty} \sigma^2 T$$

μ = mean value

σ = volatility (standard deviation)

one sensible condition is $u\cdot d = 1$



same value on each horizontal line

$$\Rightarrow \mathbb{E}(Y_j) = 2np\ln u - n\ln u$$
$$= (\ln u)\,n\,(2p-1)$$

$$\mathrm{Var}(Y_j) = 4(\ln u)^2\,np(1-p)$$

several choices for u and p are possible, the most common is: $p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{n}}$

$u = e^{\sigma\sqrt{\frac{T}{n}}}$

check:

$$\Rightarrow \mathbb{E}(Y_{ij}) = \ln e^{\sigma\sqrt{\frac{T}{n}}} \, n \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}} - 1 \right)$$

$$= n \sigma \sqrt{\frac{T}{n}} \, \frac{\mu}{\sigma} \sqrt{\frac{T}{n}} = \mu T$$

$$\Rightarrow \operatorname{Var}(Y_{ij}) = 4 \left(\ln e^{\sigma\sqrt{\frac{T}{n}}} \right)^2 n \frac{1}{4} \left(1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{n}} \right) \left(1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{n}} \right)$$

$$= 4 \left(\sigma \sqrt{\frac{T}{n}} \right)^2 n \frac{1}{4} \left(1 - \frac{\mu^2}{\sigma^2} \frac{T}{n} \right)$$

$$= \sigma^2 T \left(1 - \underbrace{\frac{\mu^2}{\sigma^2} \frac{T}{n}}_{\xrightarrow{n \to \infty} 0} \right) \xrightarrow{n \to \infty} \sigma^2 T$$

note: another possibility is $p = \frac{1}{2}$

$$u = \exp\left(\mu \frac{T}{n} + \sigma \sqrt{\frac{T}{n}} \right)$$

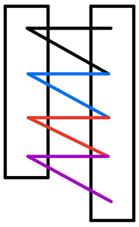
$$d = \exp\left(\mu \frac{T}{n} - \sigma \sqrt{\frac{T}{n}} \right)$$

($u \cdot d \neq 1$ here)

python implementation of binomial tree:

- to store data:
  - vectors (memory efficient)
  - matrix (if you need all data, e.g., for plots)

- for going from one column to previous one use vectorized operations
  ↳ use only one "for" loop to go through all steps



recall notation vector[a:b:increment]