Session 12
Oct. 14, 2019

last time we derived that the price of European call options can be written as

$$C = S \sum_{j=a}^{n} b(j,n,pue^{-r\frac{T}{n}}) - K e^{-rT} \sum_{j=a}^{n} b(j,n,p)$$

with $a = \frac{\ln\frac{K}{S} - n\ln d}{\ln\frac{u}{d}}$, $p = \frac{e^{r\frac{T}{n}} - d}{u-d}$

now we use the calibration $u = e^{\sigma\sqrt{\frac{T}{n}}}$, $d = \frac{1}{u}$

we find: • $a = \frac{\ln\frac{K}{S} - n\ln d}{\ln\frac{u}{d}} = \frac{\ln\frac{K}{S} - n\ln e^{-\sigma\sqrt{\frac{T}{n}}}}{2\ln e^{\sigma\sqrt{\frac{T}{n}}}} = \frac{\ln\frac{K}{S} + \sigma\sqrt{T}\sqrt{n}}{2\sigma\sqrt{\frac{T}{n}}}$

• $p = \frac{e^{r\frac{T}{n}} - d}{u-d} = \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}} = \frac{1 + r\frac{T}{n} + O(\frac{1}{n^2}) - (1 - \sigma\sqrt{\frac{T}{n}} + \frac{1}{2}\sigma^2\frac{T}{n} + O(\frac{1}{n^{3/2}}))}{1 + \sigma\sqrt{\frac{T}{n}} + \sigma^2\frac{T}{n} + O(\frac{1}{n^{3/2}}) - (1 - \sigma\sqrt{\frac{T}{n}} + \sigma^2\frac{T}{n} + O(\frac{1}{n^{3/2}}))}$

$$= \frac{\sigma\sqrt{\frac{T}{n}} + (r - \frac{\sigma^2}{2})\frac{T}{n} + O(n^{-\frac{3}{2}})}{2\sigma\sqrt{\frac{T}{n}} + O(n^{-3/2})}$$

$$= \frac{1}{2}\left(1 + \frac{(r-\frac{\sigma^2}{2})\sqrt{\frac{T}{n}}}{\sigma} + O(n^{-1})\right)$$

• $\Rightarrow \lim_{n\to\infty} \frac{a - np}{\sqrt{np(1-p)}} = \lim_{n\to\infty} \frac{\frac{\ln\frac{K}{S}}{2\sigma\sqrt{T}} n^{\frac{1}{2}} + \frac{n}{2} - \frac{n}{2} - \frac{(r-\frac{\sigma^2}{2})\sqrt{T}}{2\sigma} n^{\frac{1}{2}} + O(1)}{n^{\frac{1}{2}}\underbrace{\left[\left(\frac{1}{2} + \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)\left(\frac{1}{2} - \frac{\text{const}}{\sqrt{n}} + O(n^{-1})\right)\right]^{\frac{1}{2}}}_{=\left[\frac{1}{4} + O(n^{-1})\right]^{\frac{1}{2}} = \frac{1}{2} + O(n^{-\frac{1}{2}})}}$

$$= \frac{\ln\frac{K}{S} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Some more computations yield the following result:

Black-Scholes formula:

$$C = S\,\Phi(x) - Ke^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

with $x = \dfrac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, where $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$

(cumulative normal distribution fct.)

this concludes the chapter on discrete models

Chapter Summary:

Options: • types: – call, put

– American, European + many other types

• defined by: type, T, K, payoff

Model for stock price: here, discrete time binomial tree model $S \begin{cases} Su \\ Sd \end{cases}$

next: continuous in time geometric Brownian Motion

Option Pricing: • based on no arbitrage (no risk-free profit) and replicating portfolio

• for discrete time model: use binomial tree with backward induction

• for the special case of European calls we have a closed-form formula:

$C = e^{-rT}\,\mathbb{E}(\text{payoff})$ under binomial distribution with risk-neutral probabilities

- in the limit $n \to \infty$ this becomes Black-Scholes formula

$$C = S\,\Phi(x) - K e^{-rT}\,\Phi(x - \sigma\sqrt{T})$$

next: continuous in time models

3. Continuous Time Models

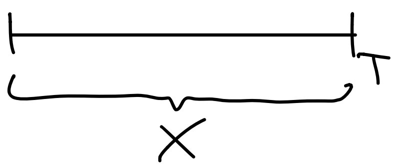
3.1 Brownian Motion

Motivation: for the binomial distribution we had the CLT:

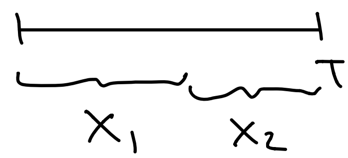
$$\sqrt{Var(j)}\; b(j;n,p) \xrightarrow{n\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

↳ normal distribution with mean 0 and variance 1 : $\mathcal{N}(0,1)$

now consider random variable X with distribution $\mathcal{N}(0,1)$:



want:



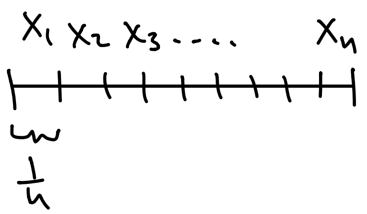
X_1, X_2 same process and independent

we have $1 = Var(X) = Var(X_1 + X_2) \underset{\text{independence}}{=} Var(X_1) + Var(X_2)$

$\underset{\text{same distribution}}{=} 2\,Var(X_1)$

$\Rightarrow X_1$ distributed according to $\frac{1}{\sqrt{2}}\mathcal{N}(0,1)$, or $\mathcal{N}(0,\frac{1}{\sqrt{2}})$.

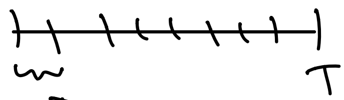
n steps:



$\Rightarrow 1 = Var(X) = n\,Var(X_1)$

$\Rightarrow X_i \sim \frac{1}{\sqrt{n}}\mathcal{N}(0,1) \quad \left(\sim \mathcal{N}(0,\frac{1}{\sqrt{n}})\right)$

or, taking T into account:



$\Delta t = \frac{T}{n}$

$\Rightarrow X_i \sim \sqrt{\Delta t}\, \mathcal{N}(0,1)$

this motivates the following rigorous definition:

<u>Def.:</u> A stochastic process $t \mapsto W(t)$ for $t \in [0,\infty)$ is called Brownian Motion (BM) or Wiener process if:

a) $W(0) = 0$

b) each realization is continuous in t

c) for any $0 \leq s_1 < s_2 < t_1 < t_2$ the increments

$W(s_2) - W(s_1)$ and $W(t_2) - W(t_1)$ are independent

d) $W(t_2) - W(t_1)$ is distributed like $\sqrt{t_2 - t_1}\, \mathcal{N}(0,1)$ for all $t_1 < t_2$

Python implementation:

- BM: $W_0 = 0$

  $W_1 = \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

  $W_2 = W_1 + \sqrt{\Delta t} \cdot$ sample from $\mathcal{N}(0,1)$

  in python: dW = normal(0, 1, size=n) $\cdot \sqrt{\Delta t}$

  W = cumsum(dW) (cumulative sum)

  W = r_[0, W] (add time 0)

  $\left(\begin{array}{l} a = (\ldots),\ b = (\ldots) \\ r\_[a,b] = (\underbrace{\ldots}_{a}, \underbrace{\ldots}_{b}) \end{array} \right)$

- ensemble of BMs: M BM paths

  in python: dW = normal(0, 1, size=(M, N))

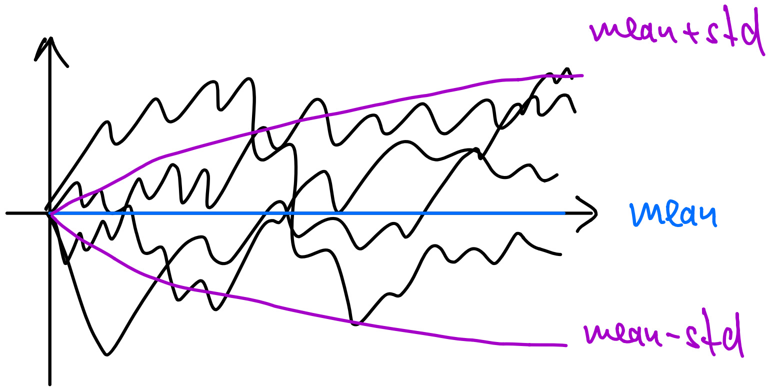
  M → # of samples, N → # of timesteps

  W = cumsum(dW, axis=1)

  ↳ cumulative sum over row entries

  e.g.: mean(W, axis=0), std(W, axis=0) (i.e., over samples)

- seed(k) for fixed k gives you same realizations


mean+std
mean
mean-std