Session 13
Oct. 17, 2019

Question from last time: Why are the probability distributions $\mathcal{N}(0,\sigma)$ and $\sqrt{\sigma}\,\mathcal{N}(0,1)$ the same?

$$P_{\mathcal{N}(0,\sigma)}(x\in[a,b]) = \int_a^b \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}\,dx$$

$$P_{\sqrt{\sigma}\mathcal{N}(0,1)}(x\in[a,b]) = P_{\mathcal{N}(0,1)}\left(x\in\frac{1}{\sqrt{\sigma}}[a,b]\right) = \int_{a/\sqrt{\sigma}}^{b/\sqrt{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$\underset{x=\frac{y}{\sqrt{\sigma}}}{=} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma}} \frac{dy}{\sqrt{\sigma}} = P_{\mathcal{N}(0,\sigma)}(x\in[a,b])$$

Last time we introduced Brownian motion (BM) $W(t)$ as a stochastic process s.t.

$W(0)=0$, W cont., increments independent, and $W(t_2)-W(t_1) \sim \sqrt{t_2-t_1}\,\mathcal{N}(0,1)$ $\forall t_1<t_2$

Note:
- BM exists and is unique
- BM is one example of a Markov process, i.e., future values are independent of current values

BM not a good model for stock prices:
- parameters mean and variance are missing
- BM can be negative

better: Geometric Brownian Motion (GBM): $S(t) = S(0)\,e^{-(\mu-\frac{\sigma^2}{2})t+\sigma W(t)}$

In the next homework we show that calibrated paths in binomial tree model converge to GBM as $n\to\infty$.

Also in next homework: powerful method to numerically evaluate expecation values

Monte-Carlo method:

random samplings to approximate expectation values

Ex.: binomial tree model for European call options:

$$C = \sum_{j=0}^{n} b(j,n,p) \underbrace{e^{-rT} \max(0, S u^j d^{n-j} - K)}_{f(j,n)} = \mathbb{E}(f)$$

Monte-Carlo: m samples $j_1, \dots, j_m$ from $b(j,n,p)$ and

compute $\frac{1}{m} \sum_{k=1}^{m} f(j_k, n) \xrightarrow{m \to \infty} \mathbb{E}(f)$

(by the (weak or strong) law of large numbers

idea/hope:
- time efficient method, since $m << n$ to yield good results
- use randomness to approximate deterministic problems

next HW problem: use GBM in Monte-Carlo method for European calls, find convergence rate.