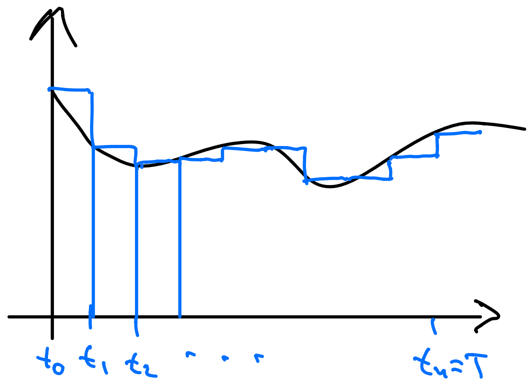
3.2 Stochastic Integrals

Session 14
Oct. 21, 2019

Recall Riemann sum for Riemann integral:

$$\int_0^T f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i$$



$$\Delta t_i = t_{i+1} - t_i = \Delta t = \frac{T}{n}$$

later: want stochastic $\underbrace{\text{PDEs}}_{\text{partial differential equation}}$ with noise: $dX = f\,dt + g\,dW$

there are different kinds of stochastic integrals

Itô-integral:

def. analogously to Riemann sum (W = Brownian motion)

$$\int_0^T f(t)\,dW(t) := \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \quad \text{with} \quad \Delta W_i = W(t_{i+1}) - W(t_i)$$

distributed like $\sim \sqrt{\Delta t}\,\mathcal{N}(0,1)$

Ex.: integrate Brownian motion against itself: $\int_0^T W(t)\,dW(t) = \int_0^T W\,dW$

If $W(t)$ were differentiable we could use the chain rule $\frac{d}{dt} f(g(t)) = f' \cdot \frac{dg}{dt}$,

i.e., here: $dW = \frac{dW}{dt} dt$

Then $\int_0^T W(t)\, dW(t) = \int_0^T W(t) \frac{dW(t)}{dt} dt = \frac{1}{2} \int_0^T \frac{d}{dt}(W(t)^2)\, dt$

$$= \frac{1}{2} W(T)^2 - \frac{1}{2} \underbrace{W(0)^2}_{=0}$$

But: BM is not differentiable! $\frac{dW}{dt}$ does not exist.

it turns out that the value of the integral is actually different:

$$\int_0^T W(t)\, dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i) \Delta W_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i)\left(W(t_{i+1}) - W(t_i)\right)}$$

$$= W(t_i) W(t_{i+1}) - W(t_i)^2$$

$$= \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 - \left(W(t_{i+1}) - W(t_i)\right)^2 \right]$$

$$\Rightarrow \int_0^T W(t)\, dW(t) = \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} \frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 \right]} - \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{\left(W(t_{i+1}) - W(t_i)\right)^2}_{?}$$

$$= \frac{1}{2} W(T)^2 - \frac{1}{2} \underbrace{W(0)^2}_{=0}$$

How is $\left[W(t_{i+1}) - W(t_i)\right]^2 = \Delta W_i^2$ distributed?

One can check that • $\mathbb{E}(\Delta W_i^2) = \Delta t = \frac{T}{n}$ ← heuristically clear from $\Delta W \sim \sqrt{\Delta t}\, N(0,1)$

✓

$$\bullet\ \mathbb{E}(\Delta W_i^4) = \Delta t^2 = \frac{T^2}{n^2}$$

so $\sum_{i=0}^{n-1} \Delta W_i^2 = \sum_{i=0}^{n-1} \left(\frac{T}{n} + o\left(\frac{1}{n^2}\right)\right) = T + O\left(\frac{1}{n}\right) \xrightarrow{n\to\infty} T$

$\Rightarrow$ in the limit $n \to \infty$, $\sum_{i=0}^{n-1} (\Delta W_i)^2$ is const! It is a deterministic process!
(the only process for which $\mathbb{E}$ is constant and all higher moments vanish)

$$\Rightarrow \int_0^T W(t)\,dW(t) = \frac{1}{2} W(T)^2 - \frac{1}{2} T$$

note: this is different from usual integral because $\Delta W \sim \sqrt{\Delta t}$ and not like Δt

<u>Stratonovich integral</u>:

this notation is sometimes used to differentiate it from Itô integral

$$\int_0^T f(t) \circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i^*)\,\Delta W_i \quad \text{with } t_i^* = \frac{t_{i+1}+t_i}{2}$$

<u>Ex.:</u> $\int_0^T W(t)\circ dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \underbrace{W(t_i^*)\left(W(t_{i+1}) - W(t_i)\right)}$

$$\frac{1}{2}\left[W(t_{i+1})^2 - W(t_i)^2 + \left(W(t_i^*) - W(t_i)\right)^2 - \left(W(t_{i+1}) - W(t_i^*)\right)^2 \right]$$

$$\Rightarrow \int_0^T W(t)\circ dW(t) = \frac{1}{2} W(T)^2 + \lim_{n\to\infty}\left[\sum_{i=0}^{n-1} \left(W(t_i^*) - W(t_i)\right)^2 - \sum_{i=0}^{n-1}\left(W(t_{i+1}) - W(t_i^*)\right)^2\right]$$

Similar to before one can compute:

$$\mathbb{E}\left(\left(W(t_i^*) - W(t_i)\right)^2\right) \sim t_i^* - t_i = \frac{t_{i+1}+t_i}{2} - t_i = \frac{t_{i+1}-t_i}{2} = \frac{\Delta t}{2}$$

and higher moments vanish if summed over similar to before

$$\Rightarrow \int_0^T W(t) \circ dW(t) = \frac{1}{2} W(T)^2 + \frac{T}{2} - \frac{T}{2} = \frac{1}{2} W(T)^2$$

In comparison:

- Stratonovich:
  - some "nicer" properties and better analogy to usual integral
  - but in each step W is evaluated in between t_i and t_{i+1}
- Itô:
  - technically a bit "harder" to handle
  - but at t_i, the increments ΔW_i are added, as we want for stock price development

next week: stochastic PDEs like $dX = f\,dt + g\,dW$, interpreted in the sense of the Itô integral