3.3 Stochastic Differential Equations

Session 16
Oct. 28, 2019

usual first order ordinary differential equation (ODE):

$$\frac{dx(t)}{dt} = f(x(t), t)$$

integral form: $x(t) = x(0) + \int_0^t f(x(s), s)\,ds$

stochastic differential equation (SDE):

→ Brownian motion increments

$$X(t) = X(0) + \int_0^t f(X(s), s)\,ds + \underbrace{\int_0^t g(X(s), s)\,dW(s)}$$

stochastic integral (always Itô from now on)

short-hand notation: $dX(t) = f(X(t), t)\,dt + g(X(t), t)\,dW(t)$

<u>Ex.:</u> $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$, $S(0) = S_0$

next week: this is solved by geom. Brownian motion

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

integral form: $S(t) - S_0 = \mu \int_0^t S(u)\,du + \sigma \int_0^t S(u)\,dW(u)$

from this we can compute $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) - S_0 = \mu \int_0^t \mathbb{E}(S(u))\,du + \sigma \int_0^t \underbrace{\mathbb{E}(S(u)\,dW(u))}_{= \mathbb{E}(S(u))\underbrace{\mathbb{E}(dW(u))}_{=0}}$$

so

$$\mathbb{E}(S(t)) = S_0 + \mu \int_0^t \mathbb{E}(S(u))\,du$$

$$\left(\frac{d\mathbb{E}(S(t))}{dt} = \mu\, \mathbb{E}(S(t)) \right)$$

$$\Rightarrow \mathbb{E}(S(t)) = e^{\mu t}$$

Usual ODE can be solved numerically with Euler's method:

discretize ODE: $\frac{x_{n+1} - x_n}{\Delta t} = f(x_n, t_n)$, $\Delta t = \frac{t}{n}$

or $x_{n+1} = x_n + f(x_n, t)\Delta t$

error in one step:

- Euler: $x_1 = x_0 + f(x_0, t)\Delta t$
- exact solution (with Taylor): $x(\Delta t) = x(0) + \Delta t \overbrace{x'(0)}^{= f(x_0, 0)} + \frac{(\Delta t)^2}{2} x''(0) + \mathcal{O}((\Delta t)^3)$

$$\Rightarrow x(\Delta t) - x_1 = \frac{(\Delta t)^2}{2} x''(0) + \mathcal{O}((\Delta t)^3) \approx c\,(\Delta t)^2$$

total error: $|x(t) - x_n| \sim c(t)\,\Delta t$, $\Delta t = \frac{t}{n}$

the generalization to SDEs is called Euler-Maruyama method:

$$X_{n+1} = X_n + f(X_n, t_n)\,\Delta t + g(X_n, t_n)\,\Delta W_n$$

for error we want to compare X_n to exact sol. $X(t)$.

one distinguishes two types of errors:

- strong error: $\mathbb{E}(|X_n - X(t)|) \sim c_s (\Delta t)^\alpha$

  α = strong order of convergence

note: relevance for individual paths via Markov's inequality

$$\mathbb{P}(|X| > a) \leq \frac{\mathbb{E}(|X|)}{a} \quad (a > 0)$$

Proof of Markov:

$$\mathbb{E}(|x|) = \int_{-\infty}^{\infty} |x| \underbrace{\rho(x)}_{\text{probability density}} dx = \int_{-a}^{a} |x| \rho(x) dx + \int_{-\infty}^{-a} |x| \rho(x) dx + \int_{a}^{\infty} |x| \rho(x) dx$$

$$\geq \int_{-\infty}^{-a} \underbrace{|x|}_{\geq a} \rho(x)\, dx + \int_{a}^{\infty} \underbrace{|x|}_{\geq a} \rho(x)\, dx$$

use $|x| > a$ in integral

$$\geq a \underbrace{\left(\int_{-\infty}^{-a} \rho(x)\, dx + \int_{a}^{\infty} \rho(x)\, dx \right)}_{\mathbb{P}(|x| > a)}$$

Then in our case:

$$\Rightarrow \mathbb{P}\left(|X_n - X(t)| > (\Delta t)^{\frac{\alpha}{2}}\right) \leq \frac{c_s(\Delta t)^{\alpha}}{(\Delta t)^{\frac{\alpha}{2}}} = c_s (\Delta t)^{\frac{\alpha}{2}}$$

probability of a large error small for individual paths

- weak error: $|\mathbb{E}(X_n) - \mathbb{E}(X(t))| \sim c_w (\Delta t)^{\beta}$, β = weak order of convergence

note: $|\mathbb{E}(X_n) - \mathbb{E}(X(t))| = |\mathbb{E}(X_n - X(t))|$

$$\leq \mathbb{E}(|X_n - X(t)|)$$

so weak error $\leq$ strong error

Ex.: compare $X(t) = 0$ to $X_n = \begin{cases} +1 & \text{with prob. } \frac{1}{2} \\ -1 & \text{with prob. } \frac{1}{2} \end{cases}$

weak error: $|0 - 0| = 0$

strong error: $\frac{1}{2} + \frac{1}{2} = 1$