Session 17
Nov. 4, 2019

3.4 Itô-lemma

first version: consider some nice fct. $h(W(t), t)$.

Goal: find a stochastic version of the chain rule

first, look at $h = h(W(t))$ (meaning $\frac{\partial h}{\partial t} = 0$)

write $h(W(t)) - h(W(0)) = \sum_{j=0}^{n-1} \left(h(W(t_{j+1})) - h(W(t_j)) \right)$

Taylor expansion:

$$h(W(t)) - h(W(0)) = \sum_{j=0}^{n-1} h'(W(t_j)) \left(W(t_{j+1}) - W(t_j) \right)$$
$$+ \sum_{j=0}^{n-1} \frac{1}{2} h''(m_j) \left(W(t_{j+1}) - W(t_j) \right)^2$$

for some $m_j = W_{s_j}$ with $s_j \in [t_j, t_{j+1}]$

now: recall $W(t_{j+1}) - W(t_j) \sim \sqrt{\Delta t}\, \mathcal{N}(0,1)$

and as before $\left(W(t_{j+1}) - W(t_j) \right)^2 \xrightarrow{n \to \infty} \Delta t$ (see also HW08)

take $\lim_{n \to \infty}$: $h(W(t)) - h(W(0)) = \int_0^t \left(\frac{\partial h}{\partial x} \right)(W(s))\, dW(s)$

$$+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 h}{\partial x^2} \right)(W(s))\, ds$$

$\Rightarrow$ in general case where $h(W(t), t)$ we have the Itô formula:

$$h(W(t),t) - h(W(0),0) = \int_0^t \left(\frac{\partial h}{\partial x}\right)(W(s),s)\, dW(s)$$
$$+ \int_0^t \left[\left(\frac{\partial h}{\partial s}\right)(W(s),s) + \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^2}\right)(W(s),s)\right] ds$$

short-hand notation: $dh = h' dW + \dot{h}\, dt + \frac{1}{2} h''\, dt$

$\left(\text{here: } h' = \frac{\partial h}{\partial x},\ \dot{h} = \frac{\partial h}{\partial t}\right)$

Ex.:

- $h(W(t),t) = W(t)^2$

  Itô: $dh = 2W dW + 0 + \frac{1}{2} 2\, dt = 2W dW + dt$

  is the SDE with solution $h = W^2$

  $\Rightarrow h(W(t)) - \underbrace{h(W(0))}_{=0} = \int_0^t 2W(s) dW(s) + \int_0^t ds$

  e.g., $\mathbb{E}\left(h(W(t))\right) = \mathbb{E}(W(t)^2) = \int_0^t 2\, \mathbb{E}(W(s)) \underbrace{\mathbb{E}(dW(s))}_{=0} + t$

- $h(W(t),t) = W(t)^4$

  $\Rightarrow dh = 4W^3 dW + 6W^2 dt$

  e.g., $\mathbb{E}(W(t)^4) = 4\int_0^t \mathbb{E}(W(s)^3)\, \mathbb{E}(dW(s)) + 6\int_0^t \mathbb{E}(W(s)^2)\, ds$

$$= \quad 0 \qquad + 6\int_0^t s\,ds$$

$$= 3t^2$$

<u>Ex.:</u> solve $dX = X^3 dt - X^2 dW$, $X(0) = 1$

write $X = h(W(t), t)$ and compare $dX = dh$

Itô: $dh = h' dW + \dot{h} dt + \frac{1}{2} h'' dt$

$\Rightarrow$ need to solve $\dot{h} + \frac{1}{2} h'' = h^3$ and $h' = -h^2$

$$\frac{dh}{dx} = -h^2 \overset{\text{separation of variables}}{\Rightarrow} \frac{dh}{-h^2} = dx \Rightarrow \int \frac{dh}{-h^2} = \int dx$$

$$\Rightarrow \frac{1}{h} = x + C$$

$$\Rightarrow h(x,t) = \frac{1}{x+C}$$ with $h(W(0),0) = h(0,0) = 1$ (initial condition)

$$\Rightarrow h(x,t) = \frac{1}{x+1}$$ (independent of t)

(check: $\dot{h} + \frac{1}{2} h'' = 0 + \frac{1}{2} \frac{2}{(x+1)^3} = \frac{1}{(x+1)^3} = h^3$ ✓)

$$\Rightarrow \text{solution } X(t) = \frac{1}{W(t)+1}$$ (note: actually blows up in finite time)

Second version: consider $dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dW(t)$

this is called an Itô process

now consider (nice) fct. $F(X(t), t)$

informally: Taylor expansion:

$$\Delta F(x,t) = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial x}\underbrace{\Delta x}_{= f\Delta t + g\Delta w} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Delta x^2 + \underbrace{\frac{1}{2}\frac{\partial^2 F}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 F}{\partial x \partial t}\Delta x \Delta t}_{\text{neglect, higher order in } \Delta t}$$

$$(\Delta x)^2 = (f\Delta t + g\Delta w)^2 = \underbrace{f^2 \Delta t^2 + fg\,\Delta t \Delta w}_{\text{neglect, higher order}} + g^2 \underbrace{\Delta w^2}_{\sim \Delta t}$$

$$\Rightarrow \Delta F = \frac{\partial F}{\partial t}\Delta t + f\frac{\partial F}{\partial x}\Delta t + g\frac{\partial F}{\partial x}\Delta w + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} g^2 \Delta t$$

Itô-lemma:

$$\boxed{dF = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} g^2\right] dt + g\frac{\partial F}{\partial x} dW}$$

note: for $X(t) = W(t)$, i.e., $f=0$, $g=1$, we get

$$dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right] dt + \frac{\partial F}{\partial x} dW$$

i.e., reduces to Itô formula from above